SREE VAHINI INSTITUTE OF SCIENCE AND TECHNOLOGY
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College Code : VHNI

DEPARTMENT OF MATHEMATICS

STUDY MATERIAL

IIB.TECH

NUMERICAL METHODS AND TRANSFORM TECHNIQUES

(SUB CODE : V231210321)

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FOREWORD

It is our distinct privilege to present this compendium of Numerical Methods and Transform Techniques specially prepared for first-year B.Tech students under the I-Scheme curriculum. Mathematics lies at the heart of all engineering disciplines; this text aims to provide a rigorous yet accessible foundation in key topics that underpin advanced courses in sciences and technology.

This volume has been structured to blend formal definitions, illustrative examples, and a diverse set of exercises to foster both conceptual understanding and problem-solving skills. The choice of topics, their organization, and the pedagogical progression have been guided by the overarching objective of aligning with the prescribed syllabus while addressing the needs of a varied student cohort.

We extend our profound gratitude to **Dr. R. Nagendra Babu, Principal**, for endorsing and supporting this endeavour; and to **Dr. K. Kiran Kumar, Dean (Academics)**, for the academic oversight, alignment with curriculum standards, and encouragement throughout the development process. We are especially thankful to **Mr. V. J. Moses, Head of Department, Basic Sciences & Humanities**, for his leadership, motivation, and valuable feedback during the preparation of this text. We are also indebted to our colleagues in the Department of Mathematics, whose peer review, suggestions, and insights have substantially enriched the clarity and correctness of the content. We likewise acknowledge the administrative and technical staff for their assistance in editing, formatting, and ensuring the timely publication of this material.

We anticipate that this text will serve as an indispensable companion to students and faculty alike, promoting a deeper appreciation of mathematical rigor and preparing students for onward challenges in their engineering journey.

Prepared by
Department of Mathematics / Course Team

Message from the Principal

Dr. R. Nagendra Babu

It gives me great pleasure to extend my greetings to all first-year B.Tech students as you embark on your engineering journey. Mathematics is a critical pillar in every branch of engineering, and in your formative years, a strong grounding in **Linear Algebra & Calculus** will equip you with the analytic tools and logical rigor needed for advanced study and innovation.

The material compiled in this text has been meticulously drafted to present concepts clearly, illustrate their relevance with examples, and offer problems that challenge and strengthen your understanding. I am confident that it will serve both as a guide and a companion through your semester, helping you grow in mathematical maturity and problem-solving ability.

I warmly acknowledge and thank the Department of Mathematics, under the leadership of **Mr. V. J. Moses, HOD, B.S. & H.**, and the academic support from **Dr. K. Kiran Kumar, Dean (Academics)**, for facilitating this work. Their commitment ensures that our students receive superior academic resources.

My hope is that as you engage with this text, you cultivate perseverance, intellectual curiosity, and discipline. I encourage you to use this as more than a textbook—as a stepping stone to critical thinking, innovation, and lifelong learning.

Wishing you success and fulfillment in your academic endeavors.

Dr. R. Nagendra Babu
Principal

Message from the Dean (Academics)

Dr. K. Kiran Kumar

I extend my warm greetings to all first-year B.Tech students as you begin this important stage of your academic journey. Mastery in mathematics furnishes an essential foundation for engineering studies, and competence in **Numerical Methods and Tramsform Techniques** in particular will support your success in diverse disciplines—be it electronics, mechanics, computer science, or civil engineering.

This course material has been carefully compiled to present topics with clarity, integrate illustrative examples, and introduce exercises that will progressively challenge your understanding and problem-solving skills. I trust that it will serve as a reliable academic tool for both students and faculty.

I also take this opportunity to commend the Department of Mathematics, led by **Mr. V. J. Moses, HOD (B.S. & H.)**, for their dedicated efforts in content development, peer review, and pedagogical alignment. Their perseverance and scholarly integrity have elevated the quality of this text.

I encourage every student to engage with the material thoroughly, attempt each exercise, and not hesitate in seeking clarification when needed. Let curiosity guide you and discipline sustain you. May this textbook be a stepping stone towards deeper learning and academic excellence.

Dr. K. Kiran Kumar
Dean (Academics)

WHY SHOULD WE LEARN NUMERICAL METHODS AND TRANSFORM TECHNIQUES?

APPLICATIONS OF NUMERICAL METHODS AND TRANSFORM TECHNIQUES

Numerical methods and transform techniques are both computational tools used to solve mathematical problems, but they approach the task differently. **Numerical methods** use approximation algorithms, like the Euler method or Runge-Kutta methods, to find approximate answers to problems that are too complex for an exact analytical solution. **Transform techniques** convert a problem from one representation to another, such as using the Fourier Transform to convert a time-domain signal to a frequency-domain signal, to make it easier to solve, often using an efficient algorithm like the Fast Fourier Transform

What it does

- **Decomposes signals**: It decomposes a complex signal into a sum of simple oscillating functions, like sines and cosines.
- **Changes domain**: It transforms a signal from its original domain (like time) into its frequency domain, where you can see the frequencies that make it up.
- **Generalizes Fourier Series**: It extends the concept of the Fourier Series to non-periodic functions, while the series is specifically for periodic functions.

How it is used

- **Signal processing**: It is fundamental to digital signal processing, where the Discrete Fourier Transform (DFT) and its efficient implementation, the Fast Fourier Transform (FFT), are used to analyze and manipulate digital signals.
- **Image processing**: It can be used to decompose an image into its frequency components, which can be useful for tasks like image filtering and compression.
- **Other applications**: It is used in various engineering and physics fields, including RADAR, to analyze and process signals.

- **These words are explains by Mr**.P.DASU, ME HoD

II B TECH NM&TT STUDY MATERIAL PREPARED BY

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MECHANICAL ENGINEERING (V23-II[nd] YEAR COURSE STRUCTURE & SYLLABUS)

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II Year I Semester

NUMERICAL METHODS AND TRANSFORM TECHNIQUES

Course Objectives:

- To elucidate the different numerical methods to solve nonlinear algebraic equations
- To disseminate the use of different numerical techniques for carrying out numerical integration.
- To furnish the learners with basic concepts and techniques at plus two level to lead them into advanced level by handling various real world applications.

Course Outcomes:

1. Evaluate the approximate roots of polynomial and transcendental equations by different algorithms. Apply Newton's forward & backward interpolation and Lagrange's formulae for equal and unequal intervals (L3)
2. Apply numerical integral techniques to different Engineering problems. Apply different algorithms for approximating the solutions of ordinary differential equations with initial conditions to its analytical computations (L3)
3. Apply the Laplace transform for solving differential equations (L3)
4. Find or compute the Fourier series of periodic signals (L3)
5. Know and be able to apply integral expressions for the forwards and inverse Fourier transform to a range of non-periodic waveforms (L3)

UNIT – I: Iterative Methods:

Introduction – Solutions of algebraic and transcendental equations: Bisection method – Secant method – Method of false position – Iteration method – Newton-Raphson method (One variable &Simultaneous Equations)

Interpolation: Newton's forward and backward formulae for interpolation –Central differences-Bassels& Stirling formulas- Interpolation with unequal intervals – Lagrange's interpolation formula

UNIT – II: Numerical integration, Solution of ordinary differential equations with initial conditions:

Trapezoidal rule– Simpson's 1/3[rd] and 3/8[th] rule– Solution of initial value problems by Taylor's series– Picard's method of successive approximations

Single step method- Euler's method –Modified Euler's method-Runge- Kutta method (second and fourth order)

Multi step method –Milne's Predictor and Corrector Method.

UNIT –III: Laplace Transforms:

Definition of Laplace transform - Laplace transforms of standard functions – Properties of Laplace Transforms – Shifting theorems–Transforms of derivatives and integrals – Unit step function – Dirac's delta function – Inverse Laplace transforms – Convolution theorem (with out proof).

Applications: Solving ordinary differential equations (initial value problems& boundary value problems) and integro differential equations using Laplace transforms.

UNIT – IV: Fourier series:

Introduction– Periodic functions – Fourier series of periodic function –Dirichlet's conditions – Even and odd functions –Change of interval– Half-range sine and cosine series.

UNIT – V: Fourier Transforms:

Fourier integral theorem (without proof) – Fourier sine and cosine integrals – Infinite Fourier transforms – Sine and cosine transforms – Properties– Inverse transforms – Convolution theorem (without proof) – Finite Fourier Cosine& sine transforms.

Text Books:

1. **B. S. Grewal,** Higher Engineering Mathematics, 44th Edition, Khanna Publishers.
2. **B. V. Ramana,**Higher Engineering Mathematics, 2007 Edition, Tata Mc. Graw Hill Education.

Reference Books:

1. **Erwin Kreyszig,**Advanced Engineering Mathematics, 10th Edition, Wiley-India.
2. **Steven C. Chapra,** Applied Numerical Methods with MATLAB for Engineering and Science,Tata Mc. Graw Hill Education.
3. **M. K. Jain, S.R.K. Iyengar and R.K. Jain,** Numerical Methods for Scientific and Engineering Computation, New Age International Publications.
4. **Lawrence Turyn,**Advanced Engineering Mathematics, CRC Press.

I Iterative Methods

4.1 INTRODUCTION

Determination of roots of an equation of the form $f(x)=0$ has great importance in the fields of science and Engineering. In this chapter we consider some simple methods of obtaining approximate roots of algebraic and transcendental equations.

Numerical methods are often, of a repetitive nature. These consist in repeated execution of the same process where at each step the result of the preceding step is used. This is known as *iteration process* and is repeated till the result is obtained to a desired degree of accuracy.

4.2 DEFINITIONS

1. Polynomial Function :

A function $f(x)$ is said to be a polynomial function if $f(x)$ is a polynomial in x.

i.e. $f(x)=a_0x^n+a_1x^{n-1}+....+a_{n-1}x+a_n$, where $a_0\neq 0$, the coefficients $a_0,a_1,....,a_n$ are real constants and n is a non-negative integer.

2. Algebraic Function :

A function which is a sum or difference or product of two polynomials is called an **algebraic function**; otherwise, the function is called a **transcendental** or **non-algebraic function.**

If $f(x)$ is an algebraic function, then the equation $f(x)=0$ is called an algebraic equation.

If $f(x)$ is a transcendental function, then the equation $f(x)=0$ is called a **transcendental equation.**

e.g: $f(x)=c_1e^x+c_2e^{-x}=0$; $f(x)=2\log x-\frac{\pi}{4}=0$; $f(x)=e^{5x}-\frac{x^3}{2}+3=0$

are examples of transcendental equations.

3. Algebraic Equation :

An algebraic equation of degree n is $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+....+a_{n-1}x+a_n=0$ where $a_0,a_1,a_2,......,a_n$ are real constants and $a_0\neq 0$.

4. Transcendental Equation :

Transcendental equations are non-algebraic equations involving transcendental functions such as exponential, logarithmic, trigonometric or hyperbolic functions.

e.g: $f(x)=c_1e^x+c_2e^{-x}=0$; $f(x)=2\log x-\frac{\pi}{4}=0$; $f(x)=e^{5x}-\frac{x^3}{2}+3=0$

are examples of transcendental equations.

5. Root of an equation :

A number α (real or complex) is called a root (or solution) of an equation $f(x)=0$ if $f(\alpha)=0$. We also say that α is a zero of the function $f(x)$. Geometrically, the roots of an equation are the abscissae of the points where the graph of $y=f(x)$ cuts the x-axis.

The roots of the equation $f(x)=0$ can be obtained by the following two methods.

4.3 ITERATIVE METHODS

In the following section of this chapter, we deal with a number of iterative methods. The basic idea behind these methods is explained here.

Suppose, we have to find a root α of the equation $f(x)=0$. Let x_0 be an approximation to α. Using x_0, we generate a sequence of numbers $x_1, x_2, \ldots$. Under certain conditions this sequence converges to the root α. The method of generating better and better approximation from an initial guess is called an **Iteration method.**

Order of Convergence :

Let $\varepsilon_i = x_i - \alpha$ be the error in the i^{th} stage. If the sequence $\{x_i\}$ converges to α, then the sequence $\{\varepsilon_i\}$ converges to 0. Suppose error ε_i is related to $\varepsilon_{i+1} = x_{i+1} - \alpha$ by a formula $|\varepsilon_{i+1}| \le k |\varepsilon_i|^p$, where k are p are costants $k>0, p \ge 1$, then we say that the convergence is of order p.

If $p=1$, the convergence is said to be **linear.**

If $p=2$, the convergence is said to be **quadratic.**

If $p=3$, the convergence is said to be **cubic.**

We can clearly see that the convergence is faster if k is small and p is large.

4.4 DIRECT METHOD

We are familiar with the solution of the polynomial equations such as linear equation $ax+b=0$, and quadratic equation $ax^2+bx+c=0$, using direct methods or analytical methods. Analytical methods for the solution of cubic and biquadratic equations are also available. However polynomial equations of degree greater than 4 are not solvable by analytical methods. Analytical methods are not useful in solving most of transcendental equations.

4.5 BISECTION METHOD

Bisection method is a simple iteration method to solve an equation. This method is also known as Bolzano method of successive bisection. Some times it is referred to as half-interval method. Suppose we know an equation of the form $f(x)=0$ has exactly one real root between two real numbers x_0, x_1. The number is choosen such that $f(x_0)$ and $f(x_1)$ will have opposite sign. Let us bisect the interval $[x_0, x_1]$ into two half intervals and find the mid point $x_2 = \frac{x_0+x_1}{2}$. If $f(x_2)=0$ then x_2 is a root. If $f(x_1)$ and $f(x_2)$ have same sign then the root lies between x_0 and x_2. The interval is taken as $[x_0, x_2]$. Otherwise the root lies in the interval $[x_2, x_1]$.

Repeating the process of bisection, we obtain successive subintervals which are smaller. At each iteration, we get the mid-point as a better approximation of the root. This process is terminated when interval is smaller than the desired accuracy. This is also called as "Interval Halving method".

Convergence of Bisection Method :

In the above process, let x_m be the iterative value and α is the root, then the length of the interval which contains x_m and α is $\frac{b-a}{2^m}$.

$$i.e., \ |x_m - \alpha| \le \frac{b-a}{2^m} \quad \text{.... (1)}$$

As $m \to \alpha, x_m \to \alpha$ proving the convergence that the sequence x_i tends to the root α.

Let ε_1 denote the error in i^{th} stage *i.e.,* $\varepsilon_i = x_i - \alpha$.

From (1), with $m = i$ and $i+1$, we obtain $|\varepsilon_{i+1}| \le \frac{1}{2}|\varepsilon_i|$.

Hence the convergence of the Bisection method is linear.

Note : If the error is to be made less than a small $+ve$ quantity δ, from (1), we find

$$\frac{b-a}{2^m} < \delta \ \text{(or)} \ m > \frac{ln\left(\frac{b-a}{\delta}\right)}{ln\,2}.$$

This formula is useful for determining the number of bisections needed to achieve a desired accuracy.

Merits and Demerits of Bisection method :

1. The formula used to compute the root is very simple.
2. The method converges definitely but the convergence is linear.
3. The method requires two starting values.

SOLVED EXAMPLES

Example 1 : Find a root of the equation $x^3 - 5x + 1 = 0$ using the Bisection method in 5 stages. **[JNTU (A)June 2010 (Set No.1)]**

Solution : Let $f(x) = x^3 - 5x + 1$. We note that $f(0) > 0$ and $f(1) < 0$.

$\therefore$ One root lies between 0 and 1.

Consider $x_0 = 0$ and $x_1 = 1$.

By Bisection method, the next approximation is $x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2}(0+1) = 0.5$

We have $f(x_2) = f(0.5) = -1.375 < 0$ and $f(0) > 0$.

Thus the root lies between 0 and 0.5. Now $x_3 = \frac{0+0.5}{2} = 0.25$

We find $f(x_3) = -0.234375 < 0$ and $f(0) > 0$.

Since $f(x_3) < 0$ and $f(0) > 0$, we conclude that root lies between x_0 and x_3.

The third approximation of the root is $x_4 = \frac{x_0 + x_3}{2} = \frac{1}{2}(0+.25) = 0.125$

We have $f(x_4) = 0.37695 > 0$. Since $f(x_4) > 0$ and $f(x_3) < 0$, the root lies between $x_4 = 0.125$ and $x_3 = 0.25$.

The 4^{th} approximation of the root is $x_5 = \frac{x_3 + x_4}{2} = \frac{1}{2}(0.125 + .25) = 0.1875$.

Now $f(x_5) = 0.06910 > 0$. Since $f(x_5) > 0$ and $f(x_3) < 0$, the root must lie between $x_5 = 0.1875$ and $x_3 = 0.25$.

Here the fifth approximation of the root is $x_6 = \frac{1}{2}(x_5 + x_3) = \frac{1}{2}(0.1875 + 0.25) = 0.21875$

We are asked to do upto 5 stages.

Hence we stop here. 0.21875 is taken as an approximate value of the root and it lies between 0 and 1.

Example 2 : Find a positive root of $x^3 - x - 1 = 0$ correct to two decimal places by bisection method. **[JNTU (A) June 2009, (K), (A) June 2010, June 2011, (A) May 2012 (Set No. 1)**

Solution : Let $f(x) = x^3 - x - 1 = 0$.

Consider $x_0 = 1$ and $x_1 = 2 \Rightarrow f(1) = -1 < 0, \quad f(2) = 5 > 0$

$\therefore$ One root lies between 1 and 2.

By bisection method, the next approximation is, $x_2 = \frac{x_0 + x_1}{2} \Rightarrow x_2 = \frac{1+2}{2} = 1.5$

$f(1.5) = 0.875 > 0$

$\therefore$ The root lies between 1 and 1.5

Now $x_3 = \frac{x_1 + x_2}{2} = \frac{1+1.5}{2} = 1.25, f(1.25) = -0.2968$

$\therefore f(1.25) < 0$. Since $f(x_2) > 0$ and $f(x_3) < 0$, the root lies between 1.25 and 1.5

Now $x_4 = \frac{x_2 + x_3}{2} = \frac{1.25 + 1.5}{2} = 1.375$ and $f(1.375) = 0.224 > 0$

Thus the root lies between 1.25 and 1.375.

Now $x_5 = \frac{x_3 + x_4}{2} = \frac{1.25 + 1.375}{2} = 1.3125$

$\therefore f(1.3125) = -0.0515 \Rightarrow f(1.3125)$ is negative.

Now, the root lies between 1.3125 and 1.375.

So $x_6 = \dfrac{x_4 + x_5}{2} = \dfrac{1.375 + 1.3125}{2} = 1.34375$

$\therefore f(1.34375) = 0.0826 \Rightarrow f(1.34375)$ is positive.

Now the root lies between 1.3125 and 1.34375.

So $x_7 = \dfrac{x_5 + x_6}{2} = \dfrac{1.3125 + 1.34375}{2} = 1.3281$

$\therefore f(1.3281) = 0.01447 \Rightarrow f(1.3281)$ is positive

Now the root lies between 1.3125 and 1.3281. So $x_8 = \dfrac{x_5 + x_7}{2} = \dfrac{1.3125 + 1.3281}{2} = 1.32$

$\therefore f(1.32) = -0.0187$. Hence, the root is 1.32.

Example 3 : Find out the square root of 25 given $x_0 = 2.0$, $x_1 = 7.0$ using Bisection method. **[JNTU 2000S, (A)June 2010, June 2011 (Set No. 2), Dec 2011]**

Solution : Let $f(x) = x^2 - 25 = 0$. Given $x_0 = 2.0$ and $x_1 = 7.0$

By Bisection method, the next approximation is

The next approximation x_2 is $x_2 = \dfrac{x_0 + x_1}{2} = \dfrac{2+7}{2} = 4.5$

$\therefore f(x_2) = f(4.5) = x^2 - 25 = (4.5)^2 - 25 = -4.75 < 0$

Now the root lies between 4.5 and 7 .

$\therefore$ The next approximation to the root is $x_3 = \dfrac{4.5+7}{2} = 5.75$

Now, $f(5.75) = 8.0625 > 0$

Thus the root lies between 4.5 and 5.75. So $x_4 = \dfrac{4.5+5.75}{2} = 5.125$

Now $f(5.125) = 1.265625 > 0$

$\therefore$ The root lies between 4.5 and 5.125

Now $x_5 = \dfrac{4.5+5.125}{2} = 4.8125$ and $f(4.8125) = -1.839844 < 0$

$\therefore$ The root lies between 4.8125 and 5.125

Now $x_6 = \dfrac{4.8125+5.125}{2} = 4.96875$ and $f(4.96875) = -0.31152 < 0$.

$\therefore$ The root lies between 4.96875 and 5.125

Now $x_7 = \dfrac{4.96875+5.125}{2} = 5.0468$ and $f(5.0468) = 0.47019 > 0$.

$\therefore$ The root lies between 4.96875 and 5.0468

$x_8 = \dfrac{4.96875+5.0468}{2} = 5.0077$ and $f(5.0077) = 0.0778 > 0$

$\therefore$ The root lies between 4.96875 and 5.0077

$x_9 = \dfrac{4.96875+5.0077}{2} = 4.988225$ and $f(4.988225) = -0.1176 < 0$

$\therefore$ The root lies between 4.988225 and 5.0077.

$x_{10} = \dfrac{4.988225+5.0077}{2} = 4.9979625$ and $f(4.9979625) = -0.0203 < 0$

$\therefore$ The root lies between 4.9979625 and 5.0077.

$x_{11} = \dfrac{4.9979625+5.0077}{2} = 5.00283$ and $f(5.00283) = 0.0283$

$\therefore$ Square root of 25 is 5.

Example 4 : By using bisection method, find an approximate root of the equation $\sin x = \dfrac{1}{x}$ that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computation upto 7[th] stage. **[JNTU(A) June 2010 (Set No.4)]**

Solution : The given equation may be rewritten as $f(x) = 0$, where $f(x) = x\sin x - 1$

We find that $f(1) = -0.158529 < 0$ and $f(1.5) = 0.496242 > 0$. Take $x_0 = 1$ and $x_1 = 1.5$

Since $f(1)$ and $f(1.5)$ are of opposite signs, therefore, one root lies between $x_0 = 1$ and $x_1 = 1.5$.

The first approximation of this root is $x_2 = \dfrac{1}{2}(1+1.5) = 1.25$.

We find that $f(x_2) = 0.186231 > 0$.

Since $f(x_0) < 0$ and $f(x_2) > 0$, the root lies between $x_0 = 1$ and $x_2 = 1.25$.

The second approximation of this root is $x_3 = \dfrac{1}{2}(1+1.25) = 1.125$

We find that $f(x_3) = 0.015051 > 0$.

Since $f(x_0) < 0$ and $f(x_3) > 0$, the root lies between $x_0 = 1$ and $x_3 = 1.125$.

The third approximation of the root is $x_4 = \dfrac{1}{2}(1+1.25) = 1.0625$

We find that $f(x_4) = -0.071827 < 0$.

Since $f(x_4) < 0$ and $f(x_3) > 0$, the root lies between $x_4 = 1.0625$ and $x_3 = 1.125$.

The fourth approximation of the root is $x_5 = \frac{1}{2}(10625 + 1.125) = 1.09375$

We find that $f(x_5) = -0.028362 < 0$.

Since $f(x_5) < 0$ and $f(x_3) > 0$, the root lies between $x_5 = 1.09375$ and $x_3 = 1.125$.

The fifth approximation of the root is $x_6 = \frac{1}{2}(1.09375 + 1.125) = 1.109375$

We find that $f(x_6) = -0.006643 < 0$.

Since $f(x_6) < 0$ and $f(x_3) > 0$, the root lies between $x_6 = 1.109375$ and $x_3 = 1.125$.

The sixth approximation of the root is $x_7 = \frac{1}{2}(1.109375 + 1.125) = 1.1171875$

We find that $f(x_7) = 0.004208 > 0$.

Since $f(x_6) < 0$ and $f(x_7) > 0$, the root lies between $x_6 = 1.109375$ and $x_7 = 1.1171875$.

The seventh approximation of the root is $x_8 = \frac{1}{2}(1.109375 + 1.1171875) = 1.11328125$.

This is an approximate value (of the desired order) of the root of the given equation that lies between 1 and 1.5.

Example 5 : Find a positive root of the equation $x^3 - 4x - 9 = 0$ using bisection method in four stages. [JNTU (K) Nov. 2009S (Set No. 1)]

(OR) Using Bisection method, find a real root of $x^3 - 4x - 9 = 0$ upto 3 approximations. [JNTU (K) May 2016 (Set No. 3)]

Solution : Let $f(x) = x^3 - 4x - 9$. We note that $f(2) < 0$ and $f(3) > 0$.

$\therefore$ One root lies between 2 and 3. Let $x_0 = 2$ and $x_1 = 3$.

By Bisection method, the first approximation to the desired root is given by

$x_2 = \frac{x_0 + x_1}{2} = 2.5$. Now $f(x_2) = f(2.5) = -3.375 < 0$.

$\therefore$ The root lies between x_2 and x_1.

The second approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2) = \frac{2.5 + 3}{2} = 2.75$.

Now $f(x_3) = f(2.75) = 0.7969 > 0$.

$\therefore$ The root lies between x_2 and x_3.

Thus the third approximtion to the root is $x_4 = \frac{1}{2}(x_2 + x_3) = 2.625$

Again $f(x_4) = f(2.625) = -1.4121 < 0$. $\therefore$ The root lies between x_3 and x_4.

Fourth approximation is $x_5 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(2.75 + 2.625) = 2.6875$

Example 6 : Find a real root of the equation $x \log_{10} x = 1.2$ which lies between 2 and 3 by bisection method. [JNTU Sup 2008 (Set No.

Solution : Given function is

$$f(x) = x\log_{10} x - 1.2 = 0.$$

We have $f(2) = -0.59 < 0$ and $f(3) = 0.23 > 0$

Since $f(2) < 0, f(3) > 0$, the root lies between 2 and 3.

The First approximation to the required root $x_1 = \frac{2+3}{2} = 2.5$

$f(2.5) = -0.205 < 0$ and $f(3) = 0.23 > 0$

Thus the root lies between 2.5 and 3.

The Second approximation to the required root is $x_2 = \frac{2.5+3}{2} = 2.75$

We have, $f(2.75) = 0.008164 > 0$. Also $f(2.5) < 0$, so the root lies between 2.75 and 2.5

∴ The Third approximation to the required root is

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

Now $f(2.625) = -0.997 < 0$ and $f(2.75) = 0.08164 > 0$

∴ The Fourth approximation to the required root is

$$x_4 = \frac{2.625+2.75}{2} = 2.6875.$$

Now $f(2.6875) = -0.046 < 0$ and $f(2.75) = 0.08164 > 0.$

∴ The Fifth approximation to the required root is

$$x_5 = \frac{2.6875+2.75}{2} = 2.7188$$

Now $f(2.7188) = -0.019 < 0$

∴ The sixth approximation is $x_6 = \frac{2.75+2.7188}{2} = 2.7344$

Now $f(2.7344) = -0.005 < 0$

∴ The seventh approximation to the root is given by

$$x_7 = \frac{2.75+2.7344}{2} = 2.7227$$

Now $f(x_7) = -0.015 < 0$

Hence the desired root is 2.7227.

Example 7 : Find a real root of the equation $x^3 - 6x - 4 = 0$ by bisection method.

[JNTU May 2006 (Set No.1)]

(OR) Using Bisection method, find a real root of $x^3 - 6x - 4 = 0$ upto 3 approximations.

[JNTU (K) May 2016 (Set No. 2)]

Solution : Let $f(x) = x^3 - 6x - 4$. Then

$f(2) = 8 - 12 - 4 = -8 < 0$ and $f(3) = 27 - 18 - 4 = 5 > 0$

$\therefore$ One root lies between 2 and 3.

Consider $x_0 = 2$ and $x_1 = 3$.

By Bisection method, the next approximation is $x_2 = \dfrac{x_0 + x_1}{2} = 2.5$

We have $f(x_2) = f(2.5) = -3.375 < 0$ and $f(3) > 0$.

$\therefore$ The root lies between 2.5 and 3.

Take $x_3 = \dfrac{2.5 + 3}{2} = \dfrac{5.5}{2} = 2.75$

Now $f(2.75) = 0.2968 > 0$ and $f(2.5) < 0$.

$\therefore$ The root lies between 2.5 and 2.75.

Next approximation is $x_4 = \dfrac{2.5 + 2.75}{2} = \dfrac{5.25}{2} = 2.625$

Now $f(2.625) = -1.6621 < 0$ and $f(2.75) > 0$.

$\therefore$ The root lies between 2.625 and 2.75.

Take $x_5 = \dfrac{2.625 + 2.75}{2} = 2.6875$

Now $f(x_5) = f(2.6875) = -0.7141 < 0$ and $f(2.75) > 0$.

$\therefore$ The root lies between 2.6875 and 2.75

Next approximation is $x_6 = \dfrac{2.6875 + 2.75}{2} = 2.71875$

This is the approximate value of the root of the given equation.

Example 8 : Find a real root of $x^3 - 5x + 3 = 0$ using bisection method.

[JNTU May 2006 (Set No. 2)]

(OR) Using Bisection method, find a real root of $x^3 - 5x + 3 = 0$ upto 3 approximations.

[JNTU (K) May 2016 (Set No. 4)]

Solution : Given equation is $x^3 - 5x + 3 = 0$

We have $f(2) = 8 - 10 + 3 = 1 > 0$ and $f(1) = 1 - 5 + 3 = -1 < 0$.

$\therefore$ The root lies between 1 and 2.

Let $x_0 = 0$ and $x_1 = 2$. Then the first approximation to the desired root is given by

$x_2 = \dfrac{x_0 + x_1}{2} = \dfrac{1 + 2}{2} = 1.5$

Now $f(1.5) = (1.5)^3 - 5(1.5) + 3 = -1.125 < 0$.

Since $f(1.5) < 0$ and $f(2) > 0$, the root lies between 1.5 and 2.

Next approximation to the root is $x_3 = \frac{1.5+2}{2} = \frac{3.5}{2} = 1.75$

Now $f(1.75) = -0.3906 < 0$ and $f(2) > 0$.

$\therefore$ The root lies between 1.75 and 2.

The next approximation to the root is given by

$$x_4 = \frac{1.75+2}{2} = \frac{3.75}{2} = 1.875$$

Now $f(1.875) = 0.2167 > 0$ and $f(1.75) < 0$

Thus the root lies between 1.75 and 1.875.

The next approximation is given by

$$x_5 = \frac{1.75+1.875}{2} = 1.8125$$

This gives the approximate value of the root.

Example 9 : Find a real root of equation $x^3 - x - 11 = 0$ by bisection method.

[JNTU April 2007 (Set No.2)]

(OR) Using Bisection method, find a real root of $x^3 - x - 11 = 0$ upto 3 approximations

[JNTU (K) May 2016 (Set No.1)]

Solution : Let $f(x) = x^3 - x - 11 = 0$. Consider $x_0 = 2, x_1 = 3$

Now $f(2) = -5 < 0, f(3) = 13 > 0$. $\therefore$ One root lies between 2 and 3.

By Bisection method, the next approximation is $x_2 = \frac{x_0 + x_1}{2} = \frac{2+3}{2} = 2.5$

Now $f(2.5) = 2.125 > 0$.

$\therefore$ The root lies between 2 and 2.5.

Now $x_3 = \frac{2+2.5}{2} = 2.25$. $\therefore f(2.25) = -1.8593 < 0$

Now the root lies between 2.25 and 2.5.

Take $x_4 = \frac{2.25+2.5}{2} = 2.375$. $\therefore f(2.375) = 0.02148 > 0$

Hence the root is 2.375.

Example 10 : Using bisection method, find the negative root of $x^3 - 4x + 9 = 0$

[JNTU (H) June 2009 (Set No.2)]

Solution : Given $f(x) = x^3 - 4x + 9 = 0$

Now $f(-3)=-27+12+9=-6<0$; $f(-2)=-8+8+9=9>0$

$\therefore$ One negative root lies between -3 and -2

Take $x_0=-3, x_1=\dfrac{-3+(-2)}{2}=-2.5$

Now $f(-2.5)=-15.625+10+9=19-15.625>0$

$\therefore$ The root lies between -3 and -2.5.

Let $x_2=\dfrac{-3-2.5}{2}=\dfrac{-5.5}{2}=-2.75$. Then

$f(x_2)=f(-2.75)=-20.7968+11+9=-0.7968<0$

$\therefore$ The root lies between -2.75 and -2.5

Let $x_3=\dfrac{-2.75-2.5}{2}=\dfrac{-5.25}{2}=-2.625$

Now $f(-2.625)=-18.08+10.5+9=1.42>0$

$\therefore$ The root lies between -2.75 and -2.625

Let $x_4=\dfrac{-2.75-2.625}{2}=\dfrac{-5.375}{2}=-2.6875$

Now $f(-2.6875)=-19.41+10.75+9=0.34>0$

$\therefore$ The root lies between -2.75 and -2.6875

Let $x_5=\dfrac{-2.75-2.6875}{2}=\dfrac{-5.4375}{2}=-2.71875\simeq-2.719$

Now $f(-2.719)=-20.1014+10.876+9=-20.1014+19.876<0$

$\therefore$ The root lies between -2.719 and -2.6875

Let $x_6=\dfrac{-2.719-2.687}{2}=-2.703$

Now $f(-2.703)=-19.748+10.812+9=-19.74+19.812>0$

$\therefore$ The root lies between -2.719 and -2.703

$x_7=\dfrac{-2.719-2.703}{2}=-2.711$

Thus the approximate root is -2.711

Example 13 : Find a real root of the equation $x^3 - x - 4 = 0$ by bisection metho

[JNTU (H) June 2010 (Set N

Solution : Let $f(x) = x^3 - x - 4$

Take $x_0 = 1$ and $x_1 = 2$

We have $f(x_0) = f(1) = -4 < 0$ and $f(x_1) = f(2) = 2 > 0$

$\therefore$ The root lies between x_0 and x_1.

By Bisection method, $x_2 = \dfrac{x_0 + x_1}{2} = \dfrac{1+2}{2} = 1.5$

Now $f(x_2) = f(1.5) = 3.375 - 1.5 - 4 = -2.125 < 0$

Since $f(x_2) < 0$ and $f(x_1) > 0$, the root lies between x_1 and x_2.

$$x_3 = \frac{x_1 + x_2}{2} = \frac{2+1.5}{2} = \frac{3.5}{2} = 1.75$$

Now $f(x_3) = f(1.75) = 5.35975 - 1.75 - 4 = -0.3906 < 0$

Since $f(x_1) > 0$ and $f(x_3) < 0$, the root lies between x_1 and x_3.

$$\therefore \quad x_4 = \frac{x_1 + x_3}{2} = \frac{2+1.75}{2} = \frac{3.75}{2} = 1.875$$

Now $f(x_4) = 6.5917 - 1.875 - 4 = 0.7167 > 0$

Since $f(x_4) > 0$ and $f(x_3) < 0$, the root lies between x_3 and x_4.

$$\therefore \quad x_5 = \frac{x_3 + x_4}{2} = \frac{1.75+1.875}{2} = \frac{3.625}{2} = 1.8125$$

Now $f(x_5) = f(1.8125) = 5.9494 - 5.8125 = 0.2658 > 0$

Since $f(x_3) < 0$ and $f(x_5) > 0$, the root lies between x_3 and x_5.

$$\therefore \quad x_6 = \frac{1.75+1.8125}{2} = 1.7812$$

Now $f(x_6) = f(1.7812)$

$$= 5.6511 - 1.7812 - 4 = -0.1300$$

Since $f(x_5) > 0$ and $f(x_6) < 0$, the root lies between x_5 and x_6

$$\text{Now } x_7 = \frac{x_5 + x_6}{2} = \frac{1.8125 + 1.7812}{2} = \frac{3.5937}{2} = 1.7969$$

and $f(x_7) = 5.8019 - 1.7968 - 4 = 0.005070$

$\therefore x = 1.7969$ is the approximate root.

Example 14 : Find real root of the equation $3x = e^x$ by bisection method.

[JNTU (H) June 2010 (Set No. 2)]

Solution : Given equation is $3x = e^x$

Let $f(x) = e^x - 3x$. Then

$f(0) = 1 - 0 = 1 > 0$

and $f(1) = e^1 - 3 < 0$

$\therefore$ The root lies between $x_0 = 0$ and $x_1 = 1$.

Using Bisection method, $x_2 = \dfrac{0+1}{2} = 0.5$

Now $f(x_2) = e^{.5} - 3(.5) = 1.6487 - 1.5 = 0.1487 > 0$

Since $f(x_1) < 0$ and $f(x_2) > 0$, the root lies between x_1 and x_2.

The next approximation is given by

$$x_3 = \frac{x_1 + x_2}{2} = \frac{1 + .5}{2} = 0.75$$

Now $f(x_3) = e^{0.75} - 3(0.75) = 2.1170 - 2.25 = -0.1330 < 0$

Since $f(x_2) > 0$ and $f(x_3) < 0$, the root lies between x_2 and x_3.

The next approximation is given by

$$x_4 = \frac{x_2 + x_3}{2} = \frac{0.5 + 0.75}{2} = 0.625$$

Now $f(x_4) = e^{0.625} - 3(0.625) = 1.8682 - 1.8750 = -0.0068 < 0$

$\therefore$ We can take approximate root is $x = 0.625$.

Example 16 : Using bisection method find a root of $f(x) = x - \cos x = 0$

[JNTU (K) Feb, 2014 (Set No

Solution : Given function is $f(x) = x - \cos x$

When $x = 0, f(x) = 0 - \cos 0 = -1 < 0$

When $x = 1, f(x) = 1 - \cos(1) = 1 - 0.5403 = 0.4597 > 0$

$\therefore$ The root lies between 0 and 1.

Let $x_0 = 0$ and $x_1 = 1$. The first approximation to the desired root is given by

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0+1}{2} = 0.5$$

Now $f(x_2) = f(0.5) = 0.5 - \cos(0.5) = -0.3775 < 0$

$\therefore$ The root lies between 0.5 and 1. The secondd approximation is given by

$x_3 = \dfrac{0.5+1}{2} = 0.75$

Now $f(x_3) = f(0.75) = 0.75 - \cos(0.75) = 0.75 - 0.7316 = 0.0184 > 0$

So the root lies between 0.5 and 0.75

$\therefore\ x_4 = \dfrac{0.5+0.75}{2} = 0.625$

Now $f(x_4) = f(0.625) = 0.625 - \cos(0.625) = 0.625 - 0.8109 = -0.1859$

So the root lies between 0.625 and 0.75.

$\therefore\ x_5 = \dfrac{0.625+0.75}{2} = 0.6875$

Now $f(x_5) = f(0.6875) = 0.6875 - \cos(0.6875) = -0.0853 < 0$

$\therefore$ The root lies between 0.6875 and 0.75

Let $x_6 = \dfrac{0.6875+0.75}{2} = 0.7187$. Then

$f(x_6) = f(0.7185) = 0.7185 - \cos(0.7185) = -0.0340$

$\therefore$ The root lies between (0.7185) and 0.75.

Let $x_7 = \dfrac{0.7185+0.75}{2} = 0.7342$. Then

$f(x_7) = 0.7342 - \cos(0.7342) = -0.00811 < 0$

$\therefore$ The root lies between 0.7342 and 0.75.

The next approximation is given by

$x_8 = \dfrac{0.7342+0.75}{2} = 0.7421$

Now $f(x_8) = f(0.7421) = 0.7421 = -0.0054 < 0$

$\therefore$ The root lies between 0.7421 and 0.75.

The next approximation is given by

$x_9 = \dfrac{0.7421+0.75}{2} = 0.7460$

Now $f(x_9) = f(0.7460) = 0.0115 > 0$

$\therefore$ The root lies between 0.7421 and 0.7460.

The next approximation is given by

$$x_{10} = \frac{0.7460 + 7421}{2} = 0.7441$$

Now $f(x_{10}) = f(0.7441) = 0.0084 > 0$

Hence we take $x_{10} = 0.7441$ as approximate root.

4.6 FALSE POSITION METHOD (REGULA - FALSI METHOD)

In the false position method we will find the root of the equation $f(x)=0$. Consider two initial approximate values x_0 and x_1 near the required root so that $f(x)$ and $f(x_1)$ have different signs. This implies that a root lies between x_0 and x_1. The curve $f(x)$ crosses x-axis only once at the point x_2 lying between the points x_0 and x_1. Consider the point $A=(x_0, f(x_0))$ and $B=(x_1, f(x_1))$ on the graph and suppose they are connected by a straight line. Suppose this line cuts x-axis at x_2. We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2 and value x_1 is replaced by x_2 (see Fig. (1)). Otherwise the root lies between x_2 and x_1 and the value of x_0 is replaced by x_2 (see Fig.(2)).

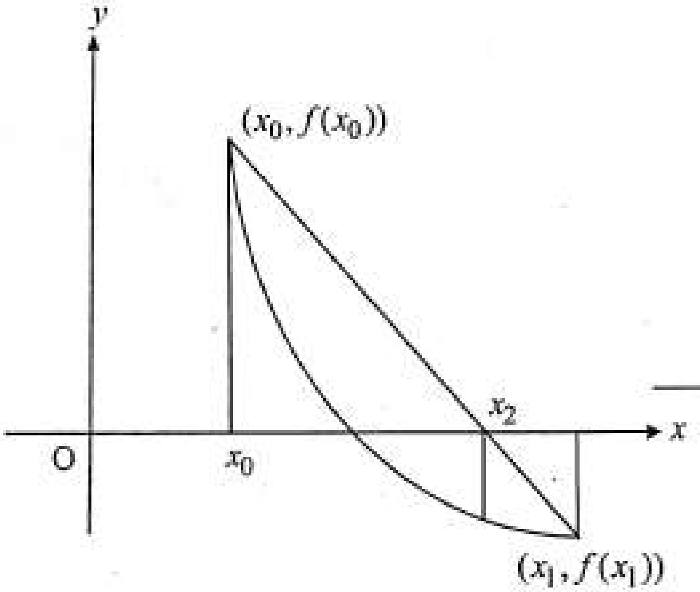

Fig.1

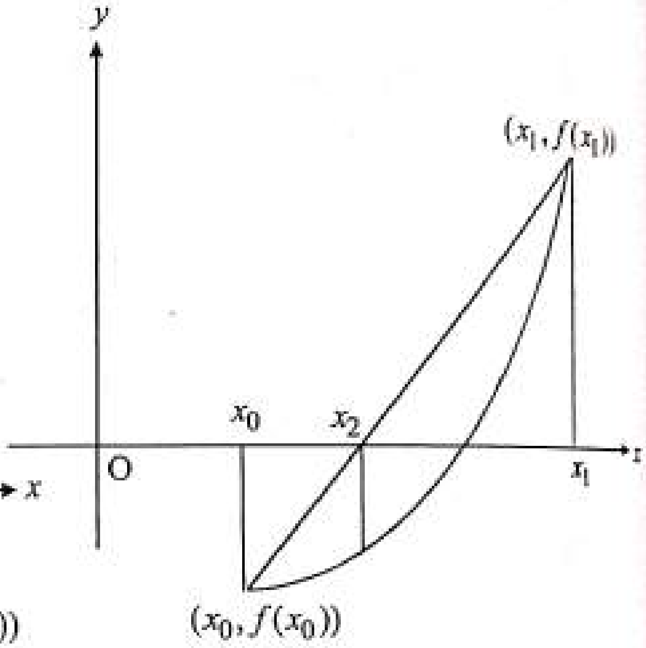

Fig.2

Another line is drawn by connecting the newly obtained pair of values. Again the point here the line cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points $x_2, x_3, x_4, \ldots$ obtained converge to the expected root of the equation $y = f(x)$.

To obtain the equation to find the next approximation to the root.

Let $A=(x_0, f(x_0))$ and $B=(x_1, f(x_1))$ be the points on the curve $y=f(x)$. Then the equation to the chord AB is $\dfrac{y-f(x_0)}{x-x_0}=\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$ .... (1)

At the point C where the line AB crosses the x-axis, we have $f(x)=0$ *i.e.* $y=0$.

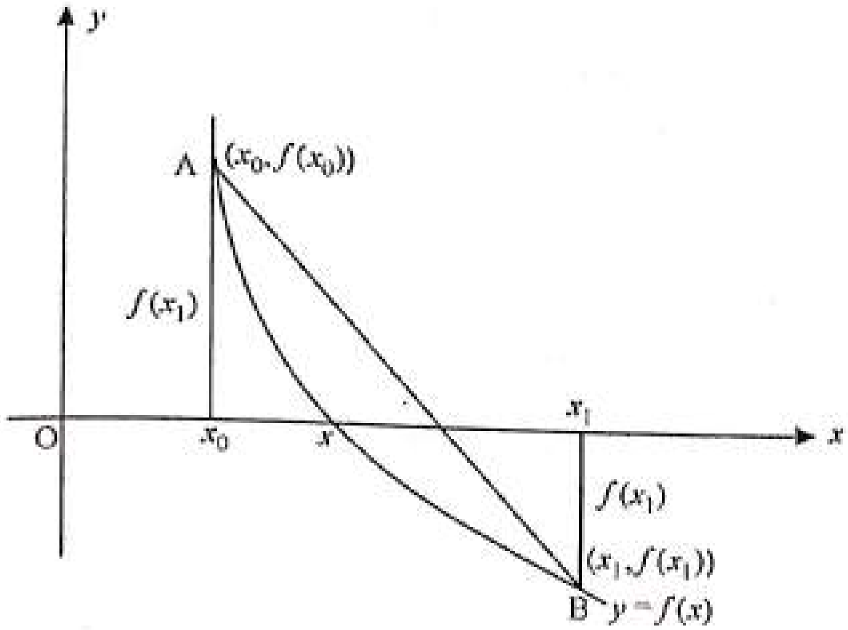

Fig.3

From (1), we get $x = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_0)$ ... (2)

x given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of x is taken as x_2 then (2) becomes

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0) = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{... (3)}$$

Now we decide whether the root lies between x_0 and x_2 or x_2 and x_1.

We name that interval as (x_1, x_2). The line joining $(x_1, y_1), (x_2, y_2)$ meets x – axis at x_3 is given by $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$.

This will in general, be nearer to the exact root. We continue this procedure till the root is found to the desired accuracy.

The iteration process based on (3) is known as the **method of False position.**

The successive intervals where the root lies, in the above procedure are named as $(x_0, x_1), (x_1, x_2), (x_2, x_3)$, etc.., where $x_i < x_{i+1}$ and $f(x_i), f(x_{i+1})$ are of opposite signs.

Also $$\boxed{x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}}$$

Merits and Demerits of Regula Falsi method :

1. The formula used to compute the root is not simple.
2. The method converges definitely and the convergence is faster than the bisection method.
3. The method requires two starting values.

SOLVED EXAMPLES

Example 1 : By using Regula-Falsi method, find an approximate root of the equation $x^4 - x - 10 = 0$ that lies between 1.8 and 2. Carry out three approximations.

[JNTU(A) June 2010 (Set No.1)]

Solution : Let us take $f(x) = x^4 - x - 10$, and $x_0 = 1.8$, $x_1 = 2$.

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$.

Since $f(x_0)$ and $f(x_1)$ are of opposite signs, the equation $f(x) = 0$ has a root between x_0 and x_1.

The first approximation of the desired root is

$$x_2 = \frac{x_0 \cdot f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.8)(4) - 2(-1.3)}{4 - (-1.3)} = \frac{7.2 + 2.6}{5.3} = \frac{9.8}{5.3} = 1.849$$

We find that $f(x_2) = -0.161$ so that $f(x_2)$ and $f(x_1)$ are of opposite signs. Hence, the root lies between x_2 and x_1 and the second order approximation of the root is

$$x_3 = \frac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)} = \frac{2(-0.161) - 1.849(4)}{-0.161 - 4} = \frac{7.7182}{4.161} = 1.8549$$

We find that $f(x_3) = f(1.8549) = -0.019$ so that $f(x_3)$ and $f(x_2)$ are of the same sign. Hence, the root doesnot lie between x_2 and x_3. But $f(x_3)$ and $f(x_1)$ are of opposite signs. So the root lies between x_3 and x_1 and the third-order approximation of the root is,

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{1.849(-0.019) - 1.8549(-0.161)}{-0.019 + 0.161} = \frac{0.2635}{0.142} = 1.8557$$

This gives the approximate value of x.

Example 2 : Find the root of the equation $x \log_{10}(x) = 1.2$ using False position method.

[JNTU Aug. 05S, 08S, (K) 09S, (A) June 10, June 11, May 2012, (K) Oct. 2018 (Set No.4)]

Solution : Let $f(x) = x \log_{10} x - 1.2$. Then

$f(2) = 2 \times \log_{10}(2) - 1.2 = 2 \times 0.30103 - 1.2 = -0.59794$

and $f(3) = 3 \times \log_{10}(3) - 1.2 = 3 \times 0.47712 - 1.2 = 0.23136$

Since $f(2)$ and $f(3)$ have opposite signs, the root lies between 2 and 3.

Take $x_0 = 2$ and $x_1 = 3$

By False position method, the first approximation of the desired root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$i.e., x_2 = \frac{2 \times 0.23136 - 3 \times (-0.59794)}{0.23136 - (-0.59794)} = 2.7210$$

Now $f(x_2) = f(2.7210) = 2.721 \times \log_{10} 2.721 - 1.2 = -0.0171 < 0$

So the root lies between 2.721 and 3.

The second approximation of the root is given by

$$x_3 = \frac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)} = \frac{2.721 \times 0.23136 - 3 \times (-0.0171)}{0.23136 - (-0.0171)} = 2.740$$

Now $f(x_3) = f(2.740) = 2.740 \times \log_{10}(2.740) - 1.2 = -0.00056 < 0$

So the root lies between 2.740 and 3.

The third approximation of the root is given by

$$x_4 = \frac{x_2 \cdot f(x_3) - x_3 \cdot f(x_2)}{f(x_3) - f(x_2)} = \frac{2.740 \times 0.23136 - 3 \times (-0.00056)}{0.23136 - (-0.00056)} = 2.7406$$

Hence the root is $x = 2.74$ (since x_3 and x_4 are same)

Example 3 : Using Regula -Falsi method, find approximate root of the equation $x^3 - x - 4 = 0$. [JNTU (A)June 2010, June 2011 (Set No. 2), Dec 2011, (K) Feb. 2015 (Set No. 4)]

Solution : Let $f(x) = x^3 - x - 4 = 0$. Then $f(0) = -4,\ f(1) = -4,\ f(2) = 2$

Since $f(1)$ and $f(2)$ have opposite signs, the root lies between 1 and 2.

Take $x_0 = 1$ and $x_1 = 2$.

By False position method, the first approximation of the desired root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad i.e.\ x_2 = \frac{(1 \times 2) - 2(-4)}{2 - (-4)} = \frac{2+8}{6} = \frac{10}{6} = 1.666$$

$\Rightarrow f(1.666) = (1.666)^3 - 1.666 - 4 = -1.042$

Now, the root lies between 1.666 and 2.

The second approximation of the root is given by

$$x_3 = \frac{1.666 \times 2 - 2 \times (-1.042)}{2 - (-1.042)} = 1.780. \quad \text{Now } f(1.780) = (1.780)^3 - 1.780 - 4 = -0.1402$$

Thus, the root lies between 1.780 and 2.

The third approximation of the root is given by

$$x_4 = \frac{1.780 \times 2 - 2 \times (-0.1402)}{2 - (-0.1402)} = 1.794. \quad \text{Now } f(1.794) = (1.794)^3 - 1.794 - 4 = -0.0201$$

Hence, the root lies between 1.794 and 2.

The next approximation to the root is given by

$$x_5 = \frac{1.794 \times 2 - 2 \times (-0.0201)}{2 - (-0.0201)} = 1.796. \quad \text{Now } f(1.796) = (1.796)^3 - 1.796 - 4 = -0.0027$$

So the root lies between 1.796 and 2.

The next approximation to the root is given by

$$x_6 = \frac{1.796 \times 2 - 2 \times (-0.0027)}{2 - (-0.0027)} = 1.796 .$$

$\therefore$ Hence the desired root is 1.796 (since x_5 and x_6 are same)

Example 4 : Find the positive root of the equation $f(x) = x^3 - 2x - 5 = 0$

[JNTU (K)Nov. 2009S (Set No.3)]

Solution : Given equation is $f(x) = x^3 - 2x - 5 = 0$

We have $f(2) = -1, f(3) = 16$. Thus, a root lies between 2 and 3.

Take $x_0 = 2$, $x_1 = 3$

We have $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2.16 - 3.(-1)}{16 + 1} = \frac{32 + 3}{17} = \frac{35}{17} = 2.059$

Again $f(x_2) = -0.386$, and hence the root lies between 2.059 and 3.

Now $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{3.(-0.386) - (2.059)(16)}{-0.386 - (16)} = 2.0812$

Repeating this process we obtain $x_4 = 2.0904$ and $x_5 = 2.0934$, etc.....

We observe that the correct value is 2.0945 and x_5 is corrected to two decimal places only. Thus it is clear that the process of convergence is very slow.

Example 5 : Find the root of the equation $2x - \log_{10} x = 7$, which lies between 3.5 and 4 by regula - falsi method. [JNTU(A) June 2010 (Set No.4)]

(or) Find a positive root of the equation $2x - \log x = 7$, by regula-false method.

[JNTU (K) May 2016 (Set No.4)]

Solution : Let $f(x) = 2x - \log_{10} x - 7 = 0$.

We have $f(3.5) = 2(3.5) - \log_{10}(3.5) - 7 = 7 - 0.5441 - 7 = -0.5441 < 0$

and $f(4) = 2(4) - \log_{10}(4) - 7 = 8 - 0.602 - 7 = 0.3979 > 0$

Since $f(3.5) < 0$ and $f(4) > 0$, the desired root lies between 3.5 and 4.

Take $x_0 = 3.5$, $x_1 = 4$.

The first approximation to the desired root by Regula-False method is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_0) = 3.5 - \frac{0.5}{0.3979 + 0.5441}(-0.5441) = 3.7888$$

Now $f(x_2) = -0.0009$, $f(x_1) = 0.3979$

$\therefore$ The root lies between 3.7888 and 4.

By taking $x_0 = 3.7888$ and $x_1 = 4$, we get the 2nd approximation of the root is given by

$$x_3 = 3.7888 - \frac{0.2112}{0.3988}(-0.0009) = 3.7893 .$$

Now $f(x_3) = 0.00004$.

Hence the required root corrected to three decimal places is 3.789. (since $f(x_3)$ is arly equal to zero).

Example 6 : Find a real root of $xe^x = 3$ using Regula - Falsi method.

[JNTU May 2006 (Set No.4)]

Solution : Let $f(x) = xe^x - 3$.

We have $f(1) = e - 3 = -0.2817 < 0$; $f(2) = 2.e^2 - 3 = 11.778 > 0$

$\therefore$ One root lies between 1 and 2.

Take $x_0 = 1$ and $x_1 = 2$.

The first approximation of the root by Regula-Falsi method is given by

$$x_2 = x_0 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) \cdot f(x_0) = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{1(11.778) - 2(-0.2817)}{11.778 + 0.2817} = 1.0234$$

Now $f(x_2) = f(1.0234) = (1.0234)\, e^{1.0234} - 3 = -0.1522 < 0$

We have $f(2) = 11.778 > 0$

$\therefore$ The root lies between 1.0234 and 2.

Taking $x_0 = 1.0234$ and $x_2 = 2$, we get the second approximation of the root is

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2(-0.522) - 1.0234(11.778)}{-0.1522 - 11.778}$$

$$= \frac{-0.3044 - 12.0536}{-11.9302} = \frac{12.358}{11.9302} = 1.0358 \approx 1.036$$

Now $f(x_3) = (1.036)\, e^{1.036} - 3 = -0.0806 < 0$

The third approximation of the root is

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = 1.043.$$ Similarly, $x_5 = 1.046$.

This gives the approximate root.

Example 8 : Find a real root of $xe^x = 2$ using Regula-falsi method.

[JNTU April 2007, (K) Feb. 2015 (Set

Solution : Let $f(x) = xe^x - 2 = 0$. Then

$f(0) = -2 < 0$; $f(1) = e - 2 = 2.7183 - 2 = 0.7183 > 0$

$\therefore$ The root lies between 0 and 1.

Take $x_0 = 0, x_1 = 1$.

By Regula - Falsi method, the first approximation of the root is

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0-(-2)}{0.7183-(-2)} = \frac{2}{2.7183} = 0.73575$$

and $f(x_2) = -0.46445 < 0$

Thus the root x_3 lies between x_1 and x_3. ($\because f(x_1) > 0$ and $f(x_2) < 0$)

The second approximation of the root is given by

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = \frac{(0.73575)\ (0.7183) - (1)\ (0.46445)}{0.7183 + 0.46445}$$

$$= \frac{0.52848 + 0.46445}{1.18275} = \frac{0.992939}{1.18275} = 0.83951$$

$\therefore$ $f(x_3) = -0.056339 < 0$

Now the root x_4 lies between x_1 and x_3. ($\because f(x_1) > 0$ and $f(x_3) < 0$)

$$\therefore x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)} = \frac{(0.83951)\ (0.7183) + 0.056339}{0.7183 + 0.056339}$$

$$= \frac{0.65935}{0.774639} = 0.851171$$

and $f(x_4) = -0.006227 < 0$

Now the root x_5 lies between x_1 and x_4.

$$\therefore x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)} = \frac{(0.851171)(0.7183) + 0.006227}{0.7183 + 0.006227}$$

$$= \frac{0.617623}{0.724527} = 0.85245$$

and $f(x_5) = -0.0006756 < 0$

Now x_6 lies between x_1 and x_5.

$$\therefore x_6 = \frac{x_5 f(x_1) - x_1 f(x_5)}{f(x_1) - f(x_5)} = \frac{(0.85245)(0.7183) + 0.0006756}{0.7183 + 0.0006756}$$

$$= \frac{0.612990}{0.71897} = 0.85260$$

and $f(x_6) = -0.00002391 < 0$

Now the root x_7 lies between x_1 and x_6.

$$\therefore x_7 = \frac{x_6 f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)} = \frac{(0.85260)(0.7183) + 0.00002391}{0.7183 + 0.00002391}$$

$$= 0.85260$$

Since $x_6 = x_7$, the desired root of $xe^x - 2 = 0$ is 0.85260.

Example 9 : Find a real root of the equation, $\log x = \cos x$ using Regula-Falsi method.

[JNTU (H) June 2011, (K) Feb. 2015, May 2016 (Set No. 1)]

Solution : Given equation is $\log x = \cos x$

Let $f(x) = \log x - \cos x$. Then

$f(1) = \log(1) - \cos(1) = 0 - 0.5403 = -0.5403 < 0$

and $f(2) = 0.6931 + 0.4161 = 1.1092 > 0$

$\therefore$ The root lies between 1 and 2.

Take $x_0 = 1$ and $x_1 = 2$.

The first approximation to the root is

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{(1)(1.1092) - (2)(-0.5403)}{1.1092 + 0.5403}$$

$$= \frac{1.1092 + 1.0806}{1.6495} = \frac{2.1898}{1.6495} = 1.3275$$

Now $f(x_2) = 0.2832 - 0.2409 = 0.0423 > 0$

$\therefore$ The root lies between x_0 and x_2

The second approximation to the root is

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = \frac{(1)(0.0423) - (1.3275)(-0.5403)}{(0.0423) + 0.5403}$$

$$= \frac{0.7595}{0.5826} = 1.3037$$

Now $f(x_3) = -0.1487 < 0$

$\therefore$ The root lies between x_3 and x_2.

The third approximation to the root is

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)}$$

$$= \frac{(1.3037)(0.0423) - (1.3275)(-0.1487)}{0.0423 + 0.1487} = \frac{(0.0551) + (0.1973)}{0.191}$$

$$= \frac{0.2524}{0.191} = 1.3214$$

Thus we take the approximate value of the root is 1.3214.

Example 10 : Find the real root of the equation $xe^x = \cos x$ using the Regula method correct to four decimal places **[JNTU (A) May 2012 (Set No. 3), (K) Dec.2**

Solution : Let $f(x) = \cos x - xe^x = 0$. We have, $f(0) = 1$ and $f(1) = -2.1779 < 0$

$\therefore$ A root lies between 0 and 1. Take $x_0 = 0$ and $x_1 = 1$.

By False method, the first approximation of the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0 - 1}{-2.1779 - 1} = 0.3146$$

Now $f(x_2) = f(0.3146) = 0.5198 > 0$

and $f(x_1) = -2.1779 < 0$

$\therefore$ The root lies between x_1 and x_2 *i.e.*, 0.3146 and 1.

The second approximation of the root is given by

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(1)(0.5198) - (0.3146)(-2.1779)}{(0.5198) + (2.1779)} = 0.4467$$

Now $f(x_3) = f(0.4467) = 0.2035 > 0$ and $f(x_1) = -2.1779 < 0$

$\therefore$ The root lies between x_1 and x_3 *i.e.*, 0.4467 and 1.

The third approximation of the root is given by

$$x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{1(0.2035) - (0.4467)(-2.1779)}{-0.2035 - 2.1779} = 0.4940$$

Continuing this process, we get

$x_5 = 0.5099$; $x_6 = 0.5152$; $x_7 = 0.5169$

$x_8 = 0.5174$; $x_9 = 0.5176$; $x_{10} = 0.5177$

Thus we will take 0.5177 as correct root.

Example 11 : Using Regula Falsi method find a real root of $f(x) = 2x^7 + x^5 + 1 = 0$ correct upto two decimal places. **[JNTU (K) Feb. 2014 (Set No. 4)]**

Solution : Let $f(x) = 2x^7 + x^5 + 1$

We have $f(-1) = -2 - 1 + 1 = -2 < 0$

and $f(0) = 1 > 0$

$\therefore$ The root lies between 0 and -1.

Take $x_0 = -1$ and $x_1 = 0$. Then $f(x_0) = -2$ and $f(x_1) = 1$

By Regula False method, the first approximation of the root is

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(-1)(1) - (0)(-2)}{(1) - (-2)} = -\frac{1}{3} = -0.3333$$

Now $f(x_2) = 0.9949 > 0$

Since $f(x_2) > 0$ are $f(x_0) < 0$, the root lies between x_2 and x_0.

Thus the second approximation of the root is

Now $$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

$$= \frac{(-1)(0.9949) - (-0.3333)(-2)}{(0.9949) + 2} = -0.5380$$

and $f(x_3) = 0.9280 > 0$.

Since $f(x_3) > 0$ and $f(x_0) < 0$, the root lies between x_3 and x_0

The third approximation of the root is

Now $$x_4 = \frac{x_0 f(x_3) - x_3 f(x_0)}{f(x_3) - f(x_0)} = \frac{(-1)(0.9280) + (0.5380)(-2)}{(0.9280) + 2} = -0.7185$$

and $f(x_4) = -2\,(0.0988) - 0.1914 + 1$

$= -(0.1976) - (0.1914) + 1$

$= 1 - 0.3890 = 0.6110$

Since $f(x_4) > 0$ and $f(x_0) < 0$, the root lies between x_4 and x_0.

$\therefore$ The fourth approximation of the root is given by

$$x_5 = \frac{x_0 f(x_4) - x_4 f(x_0)}{f(x_4) - f(x_0)} = \frac{(-1)(0.6110) + (0.7185)(-2)}{(0.6110) + 2}$$

$$= \frac{-(0.6110) - 1.4370}{2.6110} = \frac{-2.0480}{2.6110} = -0.7843$$

Example 12 : Find the real root of the equation $x + \log_{10} x - 2 = 0$ using false position method. [JNTU (K) Dec. 2016 (Set No. 1)]

Solution : Let $f(x) = x + \log_{10} x - 2$. Then

$$f(1) = 1 + \log_{10} 1 - 2 = -1 < 0$$

$$f(2) = 2 + \log_{10} 2 - 2 = 0.3010 > 0$$

Since $f(1)$ and $f(2)$ have opposite signs, the root lies between 1 and 2.

Let $x_0 = 1$ and $x_1 = 2$

By false position method, The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{1(0.3010) - 2(-1)}{0.3010 + 1} = \frac{2.3010}{1.3010} = 1.7686$$

Now $f(x_2) = 1.7686 + \log_{10}(1.7686) - 2$

$= 1.7686 + 0.2476 - 2 = 0.0162 > 0$

$\therefore$ The root lies between 1 and 1.7686

The next approximation to the root is given by

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = \frac{1(0.0162) - (1.7686)(-1)}{0.0162 + 1} = \frac{1.7848}{1.0162} = 1.7563$$

Now $f(x_3) = 1.7563 + \log_{10}(1.7563) - 2$

$= 1.7563 + 0.2445 - 2$

$= 0.0008 > 0$

$\therefore$ The root lies between 1 and 1.7563

The next approximation to the root is given by

$$x_4 = \frac{x_0 f(x_3) - x_3 f(x_0)}{f(x_3) - f(x_0)} = \frac{1(0.0008) - (1.7563)(-1)}{(0.0008) + 1}$$

$$= \frac{(0.0008)+1.7563}{1.0008} = \frac{1.7571}{1.0008} = 1.7556$$

and $f(1.7556) = 1.7556 + \log_{10}(1.7556) - 2$

$$= 1.7556 + 0.2444 - 2 = 0.00002 > 0$$

$\therefore$ The root lies between $x_0 = 1$ and $x_4 = 1.7556$.

The next approximation to the root is given by

$$x_5 = \frac{x_0 f(x_4) - x_4 f(x_0)}{f(x_4) - f(x_0)} = \frac{1(0.00002) - (1.7556)(-1)}{0.00002 + 1}$$

$$= \frac{1.7556 + 0.00002}{1.00002} = 1.7552$$

Hence the value 1.7552 can be taken as approximate root.

Example 13 : Find the Real root of the equation $4 \sin x = e^x$ using False Position method. **[JNTU (K) Dec. 2016 (Set No. 3)]**

Solution : Let $f(x) = 4 \sin x - e^x$

Now $f(0) = -1 < 0$

and $f(1) = 4\sin(1) - e^1 = 3.3658 - 2.7182 = 0.6475 > 0$

Since $f(0) < 0$ and $f(1) > 0$, therefore, one root lies between 0 and 1.

Take $x_0 = 0$ and $x_1 = 1$

First Approximation :

By Regula-Falsi method, $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$x_2 = \frac{0 - 1(-1)}{0.6475 + 1} = \frac{1}{1.6475} = 0.6069$$

Now $f(x_2) = f(0.6069) = 2.2812 - 1.8347 = 0.4464 > 0$

$\therefore$ The root lies between x_0 and x_2.

Second Approximation :

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = \frac{0 - 0.6069(-1)}{0.4464 + 1} = \frac{0.6069}{1.4484} = 0.4190$$

Now $f(0.4190) = 1.6273 - 1.5204 = 0.1068 > 0$

$\therefore$ The root lies between x_0 and x_3.

Third Approximation :

$$x_4 = \frac{x_0 f(x_3) - x_3 f(x_0)}{f(x_3) - f(x_0)} = \frac{0-(0.4190)(-1)}{0.1068+1} = \frac{0.4190}{1.1068} = 0.3785$$

Now $f(x_4) = 1.4781 - 1.4600 = 0.0181 > 0$

$\therefore$ The root lies between x_0 and x_4.

Fourth Approximation :

$$x_5 = \frac{x_0 f(x_4) - x_4 f(x_0)}{f(x_4) - f(x_0)} = \frac{0-(0.3785)(-1)}{0.0181+1} = \frac{0.3785}{1.0181} = 0.3717$$

Now $f(x_5) = f(0.3717) = 1.4527 - 1.4501 = 0.0026 > 0$

$\therefore$ The root lies between x_0 and x_5.

Fifth Approximation :

$$x_6 = \frac{x_0 f(x_5) - x_5 f(x_0)}{f(x_5) - f(x_0)} = \frac{0-(0.3717)(-1)}{0.0026+1} = \frac{0.3717}{1.0026} = 0.3707$$

Now $f(x_6) = 1.4490 - 1.4487 = 0.0003 > 0$

$\therefore$ The root lies between x_0 and x_7.

Sixth Approximation :

$$x_7 = \frac{x_0 f(x_6) - x_6 f(x_0)}{f(x_6) - f(x_0)} = \frac{0-(0.3707)(-1)}{(0.0003)+1} = \frac{0.3707}{1.0003} = 0.3705$$

$\therefore$ $x_7 = 0.3705$ can be taken as approximate root.

Example 14 : Using Regula-falsi method, find the real root of $2x - \log x = 6$ correct to three decimal places. **[JNTU (K) Dec. 2016 (MM-Set...)**

Solution : Let $f(x) = 2x - \log x - 6 = 0$

Take $x_0 = 3.5$ and $x_1 = 4$

$f(x_0) = 2 \times (3.5) - \log(3.5) - 6$

$= 1 - \log(3.5) = 1 - 0.5440 = 0.4559$

$f(x_1) = 2 \times 4 - \log(4) - 6 = 2 - \log 4 = 2 - 0.6020 = 1.397$

By Regula-Falsi Method, we have,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(3.5)(1.3979) - 4(0.4559)}{1.3979 - 0.4559}$$

$$= \frac{4.8926 - 1.8236}{0.942} = 3.2579$$

$$f(x_2) = 2 \times 3.2579 - \log(3.2579) - 6$$

$$= 6.5159 - 0.5129 - 6 = 0.5159 - 0.5129 = 0.0030$$

The root lies between x_0 and x_2.

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = \frac{(3.5)(0.0030) - 3.2579(0.4559)}{0.0030 - 0.4559}$$

$$= \frac{(0.0105) - (1.4852)}{0.4529} = \frac{-1.4747}{-0.4529} = 3.2561$$

$$f(x_3) = 2 \times 3.2561 - \log(3.2561) - 6$$

$$= 6.5122 - 0.5126 - 6$$

$$= -0.0004$$

The root lies between x_3 and x_0.

$$x_4 = \frac{x_0 f(x_3) - x_3 f(x_0)}{f(x_3) - f(x_0)} = \frac{(3.5)(-0.0004) - (3.2561)(0.4559)}{-0.0004 - 0.4559}$$

$$= \frac{-(0.0014) - (1.4844)}{-0.4563} = \frac{-1.4858}{-0.4563} = 3.2561$$

As x_3 and x_4 are coinciding we take the root as 3.2561.

Example 15 : Find a real root of the equation $x^3 - 4x - 9 = 0$ using False position method correct to three decimal places. **[JNTU (K) Dec. 2016 (MM-Set No. 3)]**

Solution : Given $f(x) = x^3 - 4x - 9 = 0$

Take $x_0 = 2.7, x_1 = 2.8$

$$f(x_0) = -0.117, f(x_1) = 1.752$$

The root lies between x_0 and x_1

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{(2.7)(1.752) - (2.8)(-0.117)}{(1.752 + 0.117)}$$

$$= \frac{(4.7304) + (0.3276)}{1.869} = 2.7062$$

$$f(x_2) = 19.8189 - 10.8248 - 9 = 19.8189 - 19.8248 = -0.0059$$

The root lies between x_1 and x_2

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(2.8)(-0.0059) - (2.7062)(1.752)}{(-0.0059) - (1.752)}$$

$$= \frac{-[0.0165 + 4.7412]}{-[1.7579]} = \frac{4.7577}{1.7579} = 2.7064$$

As x_2 and x_3 are almost equal, we take 2.7063 as the approximate root.

4.7 SECANT METHOD :

In Regula-Falsi method we are finding a real root of the equation $f(x) = 0$ approxim In that method the values of $f(x_0)$ and $f(x_1)$ are having opposite signs. In the S method we are going to discuss, the formula may be simillar, but $f(x_0)$ and $f(x_1)$ may same sign or opposite signs. In this sense, the Secant method is less restrictive than method of False position.

We want to find a real root of the equation $f(x) = 0$, approximately. Then the $y = f(x)$ meets the x-axis at the point x.

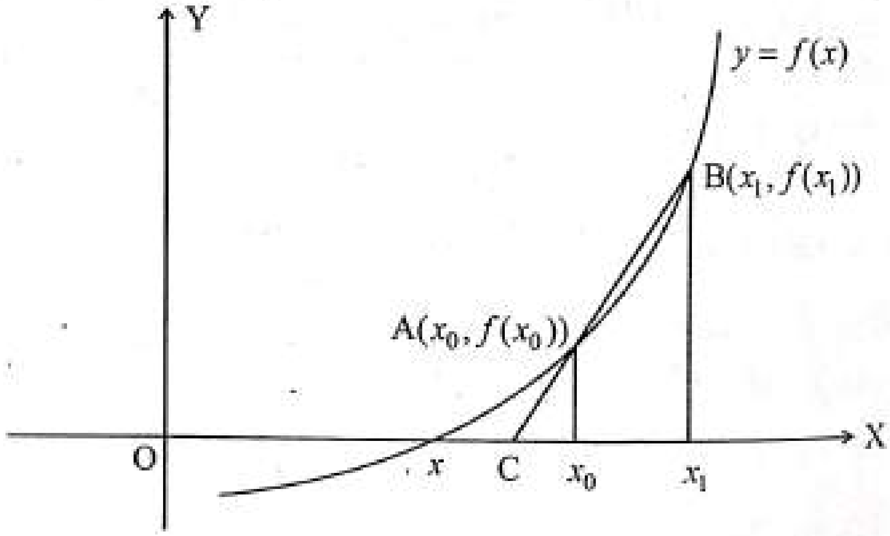

Fig.4

Consider two neighbouring points A $(x_0, f(x_0))$ and B $(x_1, f(x_1))$ on the curve $y = f$ Then the equation of the chord AB is

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad \dots (1)$$

At the point C, where the chord AB meets the X-axis, we have $f(x) = 0$ and eq (1) becomes

$$x = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} . f(x_0) \qquad \dots (2)$$

This gives the x-coordinate c of the point C. If the points A and B are sufficiently ne X-axis on the curve $y = f(x)$, the point $x = c$ is sufficiently close to the point x, wh urve meets the X-axis. This serves as an approximation for x. The method of findir pproximate value of x on the basis of this hypothesis is known as ***Secant Method.***

king Rule :

Choose an interval $[x_0, x_1]$ such that $f(x) = 0$ has exactly one root between x_0 and x_1

$x_0 < x_1$.

Apply formula (2), to find the value of x_2 of x.

$$\text{Thus, } x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

This x_2 is the x-coordinate of the point where the chord through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ meets the x-axis, it serves as the first - order approximation of x.

$$\text{Thus, } x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad \ldots (3)$$

Apply formula (2), to find the value x_3 of x.

$$\text{Thus, } x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1)$$

This x_3 determines the point where the chord through the points $(x_1, f(x_1))$ and $f(x_2))$ meets the x-axis, it serves as the second approximation.

Proceed like this to find successive approximations $x_4, x_5, \ldots\ldots$ of the root. We stop the ess when the desired order of accuracy is reached.

Example 1 : By secont method, find the root of the equation $x^3 - 5x^2 - 29 = 0$ that lies between 5 and 6, correct to four decimals.

Solution. The given equation is, $f(x) = x^3 - 5x^2 - 29 = 0$

Take $x_0 = 5$ and $x_1 = 6$.

Then $f(x_0) = f(5) = -29$ and $f(x_1) = f(6) = 7$.

Since $f(x_0)$ and $f(x_1)$ have different signs, the given equation has a root between 5 and 6.

Using Secant method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 5 - \frac{6-5}{7+29}(-29) = 5.81$$

We have $f(x_2) = f(5.81) = -1.66$

Using the formula for Secant method,

$$x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) = 6 - \frac{5.81 - 6}{-1.66 - 7} \times 7 = 5.846$$

Now $f(x_3) = -0.087$

$$\therefore\ x_4 = x_2 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_2) = 5.81 - \frac{5.846 - 5.81}{-0.087 + 1.66} \times (-1.66) = 5.84799$$

Now $f(x_4) = 0.00039$

$$\therefore x_5 = x_3 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} \times f(x_3) = 5.846 - \frac{5.84799 - 5.846}{0.00039 + 0.087} \times (-0.087) = 5.84798$$

We note that third approximation and fourth approximation of the root are identical up to the fourth place of the decimal.

Hence the approximate value of the root, correct to four decimal places is 5.8480.

Example 2. By using Secant method, find the root, correct to four places of decimal of the equation $\cos x = xe^x$, that lies between 0 and 1.

Solution. The given equation is $f(x) = \cos x - xe^x = 0$

Take $x_0 = 0$ and $x_1 = 1$. Then $f(x_0) = f(0) = 1$ and $f(x_1) = f(1) = -2.17798$

Since $f(x_0)$ and $f(x_1)$ are of opposite signs, $f(x) = 0$ has a root between 0 and 1.

The first approximation is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0 - \frac{1 - 0}{-2.17798 - 1} \times 1 = 0.31467$$

We find that $f(x_2) = f(0.31467) = 0.51987$

The second approximation of the root is

$$x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1)$$

$$= 1 - \frac{0.31467 - 1}{0.51986 + 2.17798} \times (-2.17798) = 0.44673$$

Again $f(x_3) = f(0.44673) = 0.20354$

Third approximation of the root is

$$x_4 = x_2 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_2)$$

$$= 0.31467 - \frac{0.44673 - 0.31467}{0.20354 - 0.51987} \times 0.51987 = 0.53171$$

Proceeding like this we get

$x_5 = 0.51690, x_6 = 0.51775, x_7 = 0.51776$

Since x_6 and x_7 are identical upto fourth place in decimals, we take x_7 as the root correct to four places of decimals.

$\therefore$ The root is $x_7 = 0.5178$

4.8 ITERATION METHOD

Consider an equation $f(x) = 0$ which can taken in the form $x = \phi(x)$ .... (1)

where $\phi(x)$ satisfies the following conditions:

(*i*) For two real numbers a and b, $a \le x \le b$ implies $a \le \phi(x) \le b$ and

(*ii*) For all x', x'' lying in the interval (a,b), we have $|\phi(x') - \phi(x'')| \le m|x' - x''|$ where m is a constant such that $0 \le m \le 1$.

Then, it can be proved that the equation (1) has a unique root 'α' in the interval (a,b). To find the aproximate value of this root, we start with an initial approximation x_0 of the root 'α' and find $\phi(x_0)$.

We put $x_1 = \phi(x_0)$ and take x_1 as the first approximation of α.

Next, we put $x_2 = \phi(x_1)$ and take x_2 as the second approximation of α. Continuing the process, we get the third approximtion x_3, the fourth approximation x_4, and so on.

The n^{th} approximation is given by $x_n = \phi(x_{n-1}), n \ge 1$ ..... (2)

In this process of finding successive approximations of the root α, an approximation of α is obtained by substituting the preceding approximation in the function $\phi(x)$ which is known. Such a process is called an **iteration process.** The successive approximations $x_1, x_2, \ldots$ obtained by iteration are called the **iterates.** The n^{th} approximation x_n is called the n^{th} iterate.

A formula $x_n = \phi(x_{n-1}), n \ge 1$ is called an **iterative formula.** This is also known as successive approximation method.

Convergence of an Iteration :

[redacted]

iterations are called **multi - point iterations.** The geometrical grounds in which the methods are based, indicate the iterations involved in these methods are always convergent. This fact can be proved analytically also.

Remember. Write the given equation $f(x)=0$ in the form $x=\phi(x)$

This form $x=\phi(x)$ can be choosen in many ways. We have to choose $\phi(x)$ in such a way that initial approximation x_0 should satisfy the condition $|\phi'(x_0)|<1$.

Merits and Demerits of Iteration method :

1. The formula used to find the root is very simple.
2. The method requires only one starting value.
3. The convergence of the method depends upon the starting value and the choice of $\phi(x)$.

SOLVED EXAMPLES

Example 1 : Explain the Iterative method approach in solving the problems.
[JNTU 2003, (A) June 2011 (Set No.3)]

Solution : In Latin the word iterate means to repeat. Iterative methods use a process of obtaining better and better estimates of solution with each iteration or repetitive computation. This process continues until an acceptable solution is found.

The steps involved in an iterative method are:

1. Develop an algorithm to solve a problem step-by-step.
2. Make an initial guess or estimate for the variable or variables of the solution. The initial estimates should be reasonable. Success in the solution is dependent of the selection of proper initial values of variables.
3. Better and better estimates are obtained in the progressive iterations by using the algorithm developed.
4. Stop the iteration process after reaching an acceptable solution, based on a reasonable criteria being met.

Example 2 : Explain the classification of iterative method based on the number of guesses.
[JNTU 2003]

Solution : Iterative methods can be classified into two categories based on the number of guesses:

1. Interpolation methods - also called as bracketing methods.
2. Extrapolation methods - also called as open end methods.

Two estimates are made for the root in the interpolation methods. One is positive value for the function $f(x)$ and the other gives a negative value for the function $f(x)$. This means that the root is bracketed between these two values.

Example 8 : Solve $x^3 = 2x + 5$ for a positive root by iteration method.
[JNTU 2008, (K) Oct. 2018 (Set...)]

Solution : Let $f(x) = x^3 - 2x - 5$. Then

$f(2) = (2)^3 - 2(2) - 5 = 8 - 4 - 5 = -1$

and $f(3) = (3)^3 - 2(3) - 5 = 27 - 6 - 5 = 16$

$\therefore$ One root lies between 2 and 3.

Given equation can be written as $x = \phi(x)$ in many ways such as $x^3 = 2x + 5$

$\Rightarrow \quad x = (2x + 5)^{1/3} = \phi(x)$

$$\therefore \quad \phi'(x) = \frac{2}{3}(2x+5)^{-2/3}$$

$$\Rightarrow \quad |\phi'(x)| = \left|\frac{2}{3(2x+5)^{2/3}}\right|$$

Now $|\phi'(x)|_{x=2} = \dfrac{2}{3(4+5)^{2/3}} = \dfrac{2}{3(3^{4/3})} < 1$

Hence the root lies between 2 and 3.

Also $|\phi'(x)|_{x=3} = \dfrac{2}{3(11)^{2/3}} < 1$

Now $x_0 = \dfrac{2+3}{2} = 2.5$. Then the successive approximations are

$$x_1 = \phi(x_0) = [2(2.5)+5]^{1/3} = (5+5)^{1/3} = 2.1544$$

$$x_2 = \phi(x_1) = [2(2.1544) + 5)^{1/3} = 2.1036$$

$$x_3 = \phi(x_2) = [2\,(2.1036) + 5]^{1/3} = 2.0959$$

$$x_4 = \phi(x_3) = [2(2.0959) + 5]^{1/3} = 2.09475$$

$$x_5 = \phi(x_4) = [2(2.09475) + 5]^{1/3} = 2.09458$$

and $x_6 = \phi(x_5) = [2(2.09458) + 5]^{1/3} = 2.09455$

Since $x_5 = x_6$, we conclude that the root of $x^3 - 2x - 5 = 0$ is 2.0945.

Example 9 : Find a positive root of the equation by iteration method : $3x = \cos x + 1$
[JNTU(H) June 2009 (Set...)]

Solution : Let $f(x) = \cos x - 3x + 1 = 0$

Now $f(0) = \cos 0 - 0 + 1 = 2 > 0$ and $f(\pi/2) = \cos\dfrac{\pi}{2} - 3.\dfrac{\pi}{2} + 1 = -\dfrac{3\pi}{2} + 1 < 0$

Example 8 : Solve $x^3 = 2x + 5$ for a positive root by iteration method.
[JNTU 2008, (K) Oct. 2018 (Set No.

Solution : Let $f(x) = x^3 - 2x - 5$. Then

$f(2) = (2)^3 - 2(2) - 5 = 8 - 4 - 5 = -1$

and $f(3) = (3)^3 - 2(3) - 5 = 27 - 6 - 5 = 16$

$\therefore$ One root lies between 2 and 3.

Given equation can be written as $x = \phi(x)$ in many ways such as $x^3 = 2x + 5$

$\Rightarrow \quad x = (2x+5)^{1/3} = \phi(x)$

$$\therefore \quad \phi'(x) = \frac{2}{3}(2x+5)^{-2/3}$$

$$\Rightarrow \quad |\phi'(x)| = \left|\frac{2}{3(2x+5)^{2/3}}\right|$$

$$\text{Now } |\phi'(x)|_{x=2} = \frac{2}{3(4+5)^{2/3}} = \frac{2}{3(3^{4/3})} < 1$$

Hence the root lies between 2 and 3.

$$\text{Also } |\phi'(x)|_{x=3} = \frac{2}{3(11)^{2/3}} < 1$$

Now $x_0 = \frac{2+3}{2} = 2.5$. Then the successive approximations are

$$x_1 = \phi(x_0) = [2(2.5)+5]^{1/3} = (5+5)^{1/3} = 2.1544$$

$$x_2 = \phi(x_1) = [2(2.1544)+5)^{1/3} = 2.1036$$

$$x_3 = \phi(x_2) = [2\,(2.1036)+5]^{1/3} = 2.0959$$

$$x_4 = \phi(x_3) = [2(2.0959)+5]^{1/3} = 2.09475$$

$$x_5 = \phi(x_4) = [2(2.09475)+5]^{1/3} = 2.09458$$

and $x_6 = \phi(x_5) = [2(2.09458)+5]^{1/3} = 2.09455$

Since $x_5 = x_6$, we conclude that the root of $x^3 - 2x - 5 = 0$ is 2.0945.

Example 9 : Find a positive root of the equation by iteration method : $3x = \cos x$ ...
[JNTU(H) June 2009 (Set ...

Solution : Let $f(x) = \cos x - 3x + 1 = 0$

Now $f(0) = \cos 0 - 0 + 1 = 2 > 0$ and $f(\pi/2) = \cos\frac{\pi}{2} - 3.\frac{\pi}{2} + 1 = -\frac{3\pi}{2} + 1 < 0$

$\therefore$ The root lies between 0 and $\pi/2$ (Since $f(0)>0$ and $f(\pi/2)<0$).

Given equation can be written as

$$\cos x - 3x + 1 = 0 \Rightarrow x = \frac{1}{3}(1+\cos x)$$

Let $\phi(x) = \frac{1}{3}(1+\cos x) \Rightarrow \phi'(x) = \frac{1}{3}(-\sin x)$

$|\phi'(x)| = \left|\frac{\sin x}{3}\right| < 1$ for all values of x in $(0, \pi/2)$

$\therefore$ Iteration method can be applied. Successive approximations are given by

$$x_1 = \phi(x_0) = \frac{1}{3}(1+\cos 0) = 0.6667$$

$$x_2 = \phi(x_1) = \frac{1}{3}(1+\cos 0.6667) = 0.5953$$

$$x_3 = \phi(x_2) = \frac{1}{3}(1+\cos 0.5953) = 0.60933$$

$$x_4 = \phi(x_3) = \frac{1}{3}(1+\cos 0.60933) = 0.60668$$

$$x_5 = \phi(x_4) = \frac{1}{3}(1+\cos 0.60668) = 0.60718$$

$$x_6 = \phi(x_5) = \frac{1}{3}(1+\cos 0.60718) = 0.60709$$

$$x_7 = \phi(x_6) = \frac{1}{3}(1+\cos 0.60709) = 0.60710$$

$$x_8 = \phi(x_7) = \frac{1}{3}(1+\cos 0.60710) = 0.60710$$

We observe that the iteration values x_7 and x_8 are same.

$\therefore$ The required root is 0.607.

Example 10 : Find a real root of the equation $x^3 - 2x^2 - 4 = 0$ using iteration method.

[JNTU (H) Jan. 2012 (Set No. 3)]

Solution : Given equation is $x^3 - 2x^2 - 4 = 0$ ... (1)

Now $f(2) = 8 - 8 - 4 = -4 < 0$

and $f(3) = 27 - 18 - 4 = 5 > 0$

$\therefore$ The root lies between 2 and 3.

Equation (1) can be written as $x^3 = 2x^2 + 4 \Rightarrow x = (2x^2+4)^{1/3}$

This is of the form $x = \phi(x)$; where $\phi(x) = (2x^2+4)^{1/3}$

$\therefore \quad \phi'(x) = \frac{1}{3}(2x^2+4)^{-2/3}(4x) = \frac{4x}{3(2x^2+4)^{2/3}}$

and $|\phi'(x)| < 1$ for $2 < x < 3$.

Thus iteration method can be applied.

We have

$x_0 = 2$, $\phi(x_0) = \phi(2) = (12)^{1/3} = 2.2894$

$x_1 = \phi(x_0) = 2.2894$

$x_2 = \phi(x_1) = (2x_1^2+4)^{1/3} = \left[2(2.2894)^2+4\right]^{1/3} = (14.4827)^{1/3} = 2.4375$

$x_3 = \phi(x_2) = 2(x_2^2+4)^{1/3} = \left[2(2.4375)^2+4\right]^{1/3} = (15.8830)^{1/3} = 2.5136$

$x_4 = \phi(x_3) = (16.6363)^{1/3} = 2.5528$

$x_5 = \phi(x_4) = (17.0335)^{1/3} = 2.5729$

$x_6 = \phi(x_5) = (17.2396)^{1/3} = 2.5833$

$x_7 = \phi(x_6) = (17.3468)^{1/3} = 2.5886$

$x_8 = \phi(x_7) = (17.4016)^{1/3} = 2.5913$

$x_9 = \phi(x_8) = (17.4296)^{1/3} = 2.5927$

$x_{10} = \phi(x_9) = (17.4448)^{1/3} = 2.5935$

The approximate value of the root is 2.5939

Example 11 : Find a real root of the equation $2x - \log x = 7$ using iteration meth...

[JNTU (H) Jan. 2012 (Set...]

Solution : Given equation is $f(x) = 2x - \log x - 7 = 0$

Now $f(4) = 8 - 1.3862 - 7 < 0$, $f(5) = 10 - 1.6094 - 7 > 0$.

The root lies between 4 and 5.

The given equation can be written as $2x = \log x + 7$ or $x = \frac{1}{2}(\log x + 7)$

Consider $\phi(x) = x = \frac{\log x + 7}{2}$

$\Rightarrow \phi'(x) = \frac{1}{2}\left(\frac{1}{x}\right) \quad \therefore |\phi'(x)| < 1$ in $(4,5)$

Take $x_0 = \frac{4+5}{2} = 4.5$

The first approximation of the root is given by

$$x_1 = \phi(x_0) = \phi(4.5) = \frac{\log(4.5)+7}{2} = 4.252$$

Next approximations are given by

$$x_2 = \phi(x_1) = \frac{\log(4.252)+7}{2} = 4.2236; \quad x_3 = \phi(x_2) = \frac{\log(4.2236)+7}{2} = 4.2203$$

$$x_4 = \phi(x_3) = \frac{\log(4.2203)+7}{2} = 4.2199$$

$\therefore$ We can take the approximate value of x as 4.22 (since $x_2 = x_3$).

Example 12 : Using iteration method find a real root of $f(x) = x^2 - 3x + 1$ correct upto three decimal places starting with $x = 1$. **[JNTU (K) Feb. 2014 (Set No. 3)]**

Solution : Let $f(x) = x^2 - 3x + 1 = 0$ ... (1)

Equation (1) can be written as

$$3x = x^2 + 1 \Rightarrow x = \frac{x^2+1}{3} \qquad \text{... (2)}$$

$\therefore$ Let $x = \phi(x) = \dfrac{x^2+1}{3} \Rightarrow \phi'(x) = \dfrac{2x}{3}$

We observe that $|\phi'(x)| < 1$ for $|x| < \dfrac{3}{2} \Rightarrow \dfrac{-3}{2} < x < \dfrac{3}{2}$

We have $f(0) = 1 > 0$ and $f(1) = 1 - 3 + 1 = -1 < 0$

$\therefore$ One root lies between 0 and 1.

Take $x_0 = \dfrac{0+1}{2} = 0.5$.

The first approximation of the root is

$$x_1 = \phi(x_0) = \phi(0.5) = \frac{(0.5)^2+1}{3} = \frac{0.25+1}{3} = 0.4166 \left[\because \phi(x) = \frac{x^2+1}{3}\right]$$

$$x_1 = \frac{(0.5)^2+1}{3} = \frac{0.25+1}{3} = 0.4166$$

$$x_2 = \frac{(0.4166)^2+1}{3} = 0.3911$$

$$x_3 = \frac{(0.3911)^2+1}{3} = 0.3843$$

$$x_4 = \frac{(0.3843)^2+1}{3} = 0.3825$$

$$x_5 = \frac{(0.3825)^2 + 1}{3} = 0.3821$$

$$x_6 = \frac{(0.3821)^2 + 1}{3} = 0.3820$$

Hence the desired root is taken as 0.3820.

Example 13 : Find the Real root of the equation $x^2 - x - 4 = 0$ using iteration me[thod].
[JNTU (K) Dec. 2016 (Set No.

Solution : Let $f(x) = x^2 - x - 4$

Then $f(1) = 1 - 1 - 4 = -4 < 0$

$f(2) = 4 - 2 - 4 = -2 < 0$

$f(3) = 9 - 3 - 4 = 2 > 0$

$\therefore$ One root lies between 2 and 3.

Given equation can be written as $x = \phi(x)$ in a way such that

$x^2 = x + 4 \Rightarrow x = (x+4)^{1/2} = \phi(x)$

$$\therefore \quad \phi'(x) = \frac{1}{2}\frac{1}{\sqrt{x+4}}$$

$$|\phi'(x)| = \frac{1}{2}\left|\frac{1}{\sqrt{x+4}}\right|$$

Now $|\phi'(x)|_{x=2} = \frac{1}{2}\left|\frac{1}{\sqrt{6}}\right| < 1$

and $|\phi'(x)|_{x=3} = \frac{1}{2}\left|\frac{1}{\sqrt{7}}\right| < 1$

$\therefore$ $|\phi'(x)| < 1$ for all values of x in (2,3).

Thus iteration method can be applied.

We have $x_0 = 2, x_1 = \phi(x_0) = \phi(2) = 6^{1/2} = 2.4494$

$x_1 = 2.4494, x_2 = \phi(x_1) = \sqrt{6.4494} = 2.5395$

$x_2 = 2.5395, x_3 = \phi(x_2) = \sqrt{6.5395} = 2.5572$

$x_3 = 2.5572, x_4 = \phi(x_3) = \sqrt{6.5572} = 2.5607$

$x_4 = 2.5607, x_5 = \phi(x_4) = \sqrt{6.5607} = 2.5613$

$x_5 = 2.5613, x_6 = \phi(x_5) = \sqrt{6.5613} = 2.5615$

$x_6 = 2.5615, x_7 = \phi(x_6) = \sqrt{6.5615} = 2.5615$

$x_7 = 2.5615$

We observe that $x_6 = x_7$

$\therefore$ The approximate root of the equation is 2.5615.

[illegible] root of the equation $f(x)=0$ is 0.347.

4.9 NEWTON - RAPHSON METHOD (NEWTON'S ITERATIVE METHOD)

The Newton - Raphson method is a powerful and elegent method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x)=0$, and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1)=0$. We use Taylor's theorem and expand $f(x_1) = f(x_0 + h) = 0$.

By Taylor's theorem,

$$f(x_0 + h) = f(x_0) + hf(x_0) + h^2 f''(x) + \ldots. = 0$$

$$\Rightarrow f(x_0) + h\, f'(x_0) = 0 \;\Rightarrow h = -\frac{f(x_0)}{f'(x_0)} \quad \text{(neglecting } h^2, h^3, \ldots)$$

Substituting this in x_1, we get, $x_1 = x_0 + h = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

x_1 is a better approximation than x_0.

Successive approximations are given by $x_2, x_3, \ldots, x_{n+1}$ where $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$.

This is called **Newton - Raphson formula.**

The iterative method starts with an initial approximation say x_0. Then a tangent is drawn from the corresponding point $f(x_0)$ on the curve $y = f(x)$. Let this tangent cuts the x-axis at a point say x_1 which will be a better approximation of the root. Now compute $f(x_1)$ and draw another tangent at the point $f(x_1)$ on the curve so that it cuts the x-axis at the point say x_2. The value of x_2 gives still better approximation and the process can be continued till the desired accuracy has been achieved.

Graphically this can be shown as in Fig. 4.

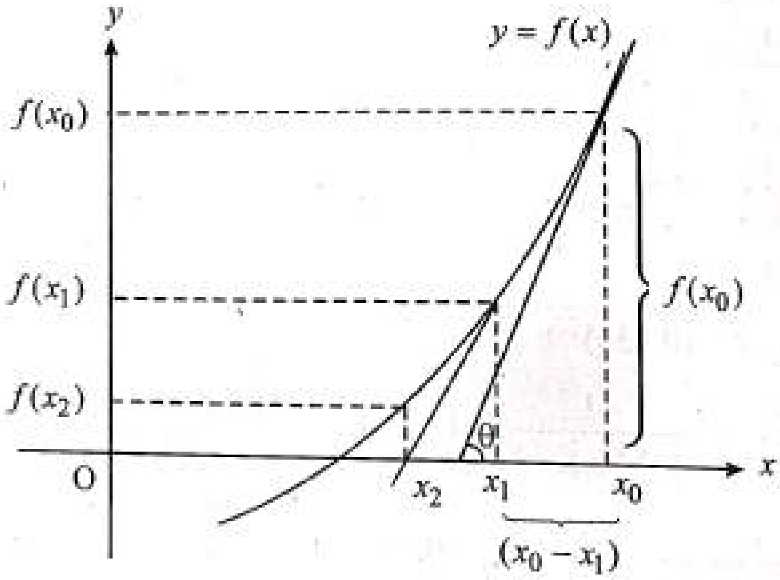

Fig. 5

1. CONVERGENCE OF NEWTON-RAPHSON METHOD

To examine the convergence of Newton-Raphson formula,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{....(1)}$$

we compare it with the general iteration formula

$$x_{i+1} = \phi(x_i)$$

where $\phi(x_i) = x_i - \dfrac{f(x_i)}{f'(x_i)}$ ...(2)

In general, we write it as

$$\phi(x) = x - \frac{f(x)}{f'(x)} \quad \text{...(3)}$$

We have already noted that the iteration method converges if $|\phi'(x)| < 1$.

$\therefore$ Newton-Raphson formula given by equation (1) converges, provided

$$|f(x) f''(x)| < |f'(x)|^2 \quad \text{...(4)}$$

In the considered interval, Newton - Raphson formula converges provided the initial approximation x_0 is chosen sufficiently close to the root and $f(x)$, $f'(x)$ and $f''(x)$ are continuous as bounded in any small interval containing the root.

2. **QUADRATIC CONVERGENCE OF NEWTON-RAPHSON METHOD [JNTU (H) 2010 (Set No. 4)]**

Suppose x_r is a root of $f(x)=0$ and x_i is an estimate of x_r such that $|x_r - x_i| = h << 1$. Then by Taylor's series expansion, we have

$$0 = f(x_r) = f(x_i + h) = f(x_i) + f'(x_i)(x_r - x_i) + \frac{f''(\xi)}{2}(x_r - x_i)^2 \text{ for some } \xi \in (x_r, x_i) \quad \dots (1)$$

By Newton-Raphson method, we know

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\Rightarrow f(x_i) = f'(x_i)(x_i - x_{i+1}) \quad \dots (2)$$

Using (2) in (1), we get

$$0 = f'(x_i)(x_r - x_{i+1}) + \frac{f''(\xi)}{2}(x_r - x_i)^2$$

Suppose $e_i = (x_r - x_i)$, $l_{i+1} = x_r - x_{i+1}$, are the errors in the solution at i^{th} and $(i+1)^{th}$ iterations.

$$\therefore \quad e_{i+1} = -\frac{f''(\xi)}{2f'(x_r)} e_i^2$$

$$\Rightarrow \quad e_{i+1} \ \alpha \ e_i^2$$

$\therefore$ The Newton method is said to have quadratic convergence.

3. **Newton-Raphson Extended Formula (or) Chebyshev's Formula of Third Order**

Theorem : The Newton-Raphson formula for finding the root of the equation $f(x)=0$

is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x)} - \dfrac{1}{2}\dfrac{[f(x_0)]^2}{[f'(x_0)]^3} f''(x_0)$ for finding the root of the equation $f(x) = 0$.

Proof : Expanding $f(x)$ by using Taylor's series and neglecting the second order terms in the neighbourhood of x_0, we obtain

$$f(x) = f(x_0) + (x - x_0) f'(x_0) \dots = 0$$

It gives $x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

This is the first approximation to the root.

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \dots (1)$$

Again expanding $f(x)$ by Taylor's series and neglecting the third order terms, we have, $f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots 0$

$$\therefore f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \frac{(x_1 - x_0)^2}{2!} f''(x_0) = 0 \quad \ldots(2)$$

Using equation (1), the equation (2) reduces to the form

$$f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2} \frac{[f(x_0)]^2}{[f'(x_0)]^2} f''(x_0) = 0$$

$\therefore$ The Newton - Raphson extended formula or Chebysher's formula of third order given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2} \frac{[f(x_0)]^2}{[f'(x_0)]^3} f''(x_0).$$

4. Merits and Demerits of Newton - Raphson Method

Merits :

1. In this method convergence is quite fast provided the starting value is close to desired root.
2. If the root is simple, the convergence is quadratic.
3. The accuracy of Newtons method for the function $f(x)$ which possess continuous first and second order derivatives can be estimated.

   If $M = \max | f''(x) |$ and $m = \min | f''(x) |$ in an interval that contains the root α and the estimator x_1 and x_2, then $| x_2 - \alpha | \le (x - \alpha)^2 . \frac{M}{m}$

   Thus the error decreases if $\left| (x_1 - \alpha)^2 . \frac{M}{m} \right| < 1$.
4. Newton - Raphson iteration is a single point iteration.
5. This method can used for solving both algebraic and transcendental equations. It can also be used when the roots are complex.

Demerits :

1. In deriving the formula for this method, it is assumed that α is a not a repeated root of $f(x) = 0$. In this case the convergence of the iteration is not guarenteed. Thus the Newton-Raphson method is not applicable to find the approximated values of repeated root.
2. Most severe limitation in the use of this method is the requirement that $f'(x) \neq 0$ in the neighbourhood of the root α. Even a moderate value of $f'(x_0)$ may move the sampled by a large value of either $f(x_0)$ or $f'(x_0)$ to produce a value x that will result in a sequence that converges to a root that we are not interested.

SOLVED EXAMPLES

Example 1 : Using Newton Raphson method find an approximate root, which lies near $x = 2$ of the equation $x^3 - 3x - 5 = 0$ upto two approximations. [JNTU (K) Oct. 2018 (Set No. 1)]

Solution : Here $f(x) = x^3 - 3x - 5 = 0$ and $f'(x) = 3(x^2 - 1)$.

By Newton-Raphson iterative formula, we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 - 3x_i - 5}{3(x_i^2 - 1)} = \frac{2x_i^3 + 5}{3(x_i^2 - 1)}, \quad i = 0, 1, 2, \ldots\ldots \quad \ldots.(1)$$

To find the root near $x = 2$, we take $x_0 = 2$. Then (1) gives

$$x_1 = \frac{2x_0^3 + 5}{3(x_0^2 - 1)} = \frac{16 + 5}{3(4 - 1)} = \frac{21}{9} = 2.3333, \quad x_2 = \frac{2x_1^3 + 5}{3(x_1^2 - 1)} = \frac{2 \times (2.3333)^3 + 5}{3\left\{(2.3333)^2 - 1\right\}} = 2.2806,$$

$$x_3 = \frac{2x_2^3 + 5}{3(x_2^2 - 1)} = \frac{2 \times (2.2806)^3 + 5}{3\left\{(2.2806)^2 - 1\right\}} = 2.2790, \quad x_4 = \frac{2 \times (2.2790)^3 + 5}{3\left\{(2.2790)^2 - 1\right\}} = 2.2790$$

Since x_3 and x_4 are identical upto 3 places of decimal, we take $x_4 = 2.279$ as the required root, correct to three places of the decimal.

Example 2 : Using the Newton-Raphson method, find the root of the equation $f(x) = e^x - 3x$ that lies between 0 and 1. [JNTU (A) June 2013 (Set No. 1)]

Solution : Here $f(x) = e^x - 3x$ and $f'(x) = e^x - 3$.

By Newton-Raphson iterative formula, we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{e^{x_i} - 3x_i}{e^{x_i} - 3} = \frac{(x_i - 1)e^{x_i}}{(e^{x_i} - 3)}, \quad i = 0, 1, 2, \ldots. \quad \ldots..(1)$$

Since the required root is supposed to lie between 0 and 1, we take x_0 to be the average of 0 and 1, *i.e.*, $x_0 = 0.5$. Then formula (1) yields

$$x_1 = \frac{((0.5) - 1)e^{0.5}}{e^{0.5} - 3} = 0.61006, \quad x_2 = \frac{(0.61006 - 1)e^{0.61006}}{e^{0.61006} - 3} = 0.618996,$$

$$x_3 = \frac{(0.618996 - 1)e^{0.618996}}{e^{0.618996} - 3} = 0.619061, \quad x_4 = \frac{(0.619061 - 1)e^{0.619061}}{e^{0.619061} - 3} = 0.619061$$

We observe that x_3 and x_4 are identical, we therefore, take $x \approx 0.619061$ as an approximate root of the given equation.

Example 3 : Using Newton-Raphson Method

(*a*) Find square root of a number

(*b*) Find the Reciprocal of a number [JNTU Sep. 2008, (K) Feb. 2015 (Set No.

(*c*) Find the m^{th} root of a number.

Solution : (*a*) Square root:

Let $f(x) = x^2 - N = 0$, where N is the number whose square root is to be found.

The solution to $f(x) = 0$ is then $x = \sqrt{N}$. Here $f'(x) = 2x$.

By Newton-Raphson technique,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2 - N}{2\,x_i} \Rightarrow x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$$

Using the above iteration formula the square root of any number N can be found to any desired accuracy. For example,

(*i*) **We will find the square root of N = 24.**

Let $= \sqrt{24}$. Then $x^2 = 24$

Let $f(x) = x^2 - 24$. Then $f'(x) = 2x$

Now $f(4) = 16 - 24 = -8 < 0$ and $f(5) = 25 - 24 = 1 > 0$

$\therefore$ The root lies between 4 and 5.

Since an approximate value of $\sqrt{24} = 4.8$, we take an initial approximation be $x_0 =$

$$x_1 = \frac{1}{2}\left(4.8 + \frac{24}{4.8}\right) = \frac{1}{2}\left(\frac{2304 + 24}{4.8}\right) = \frac{47.04}{9.6} = 4.9$$

$$x_2 = \frac{1}{2}\left(4.9 + \frac{24}{4.9}\right) = \frac{1}{2}\left(\frac{24.01 + 24}{4.9}\right) = \frac{48.01}{9.8} = 4.898$$

$$x_3 = \frac{1}{2}\left(4.898 + \frac{24}{4.898}\right) = \frac{1}{2}\left(\frac{23.9904 + 24}{4.898}\right) = \frac{47.9904}{9.796} = 4.898$$

Since $x_2 = x_3 = 4.898$, therefore, the solution to $f(x) = x^2 - 24 = 0$ is 4.898. That is the square root of 24 is 4.898.

(*ii*) **To find the square root of 10.** [JNTU Sep. 2008 (Set N

Let $x = \sqrt{10}$. Then $x^2 = 10$

Also let $f(x) = x^2 - 10 = 0$. Then $f'(x) = 2x$

Here, $a = 10$, $x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$

Now $f(3) = 9 - 10 = -1 < 0$ and $f(4) = 16 - 10 = 6 > 0$

∴ The root lies between 3 and 4.

Since an approximate value of $\sqrt{10} = 3.16$, we take an initial approximate root of the given equation as $x_0 = 3.8$. Then

$$x_1 = \frac{1}{2}\left[3.8 + \frac{10}{3.8}\right] = 3.21579 = 3.216\,;\ x_2 = \frac{1}{2}\left[3.216 + \frac{10}{3.216}\right] = 3.1627$$

$$x_3 = \frac{1}{2}\left[3.162 + \frac{10}{3.1627}\right] = 3.1627$$

Since $x_2 = x_3 = 3.162$, therefore, the solution to $f(x) = x^2 - 10 = 0$ is 3.162. Thus we can take square root of 10 as 3.1627.

(b) **Reciprocal:**

Let $f(x) = \frac{1}{x} - N = 0$ where N is the number whose reciprocal is to be found.

The solution to $f(x)$ is then $x = \frac{1}{N}$. Also, $f'(x) = \frac{-1}{x^2}$

To find the solution for $f(x) = 0$, apply Newton-Raphson technique.

$$x_{i+1} = x_i - \frac{\left(\frac{1}{x_i} - N\right)}{\frac{-1}{x_i^2}} = x_i + x_i^2\left(\frac{1}{x_i} - N\right) = x_i + x_i^2\left(\frac{1 - Nx_i}{x_i}\right)$$

$$= x_i + x_i(1 - Nx_i) = 2x_i - Nx_i^2 = x_i(2 - x_iN) \qquad \dots (1)$$

For example, the calculation of reciprocal of 22 is as follows.

Since an approximate value of $\frac{1}{22} = 0.045$, we take the initial approximation, $x_0 = 0.045$.

∴ $x_1 = x_0(2 - x_0 N)$ [Putting $i = 0$ in (1)]

$= 0.045\,(2 - 0.045 \times 22) = 0.045\,(2 - 0.99) = 0.045\,(1.01) = 0.0454$

$x_2 = 0.0454\,(2 - 0.0454 \times 22) = 0.0454\,(2 - 0.9988) = 0.0454\,(1.0012) = 0.04545$

$x_3 = 0.04545\,(2 - 0.04545 \times 22) = 0.04545\,(2 - 0.9999) = 0.04545\,(1.0001) = 0.04545$

$x_4 = 0.04545\,(2 - 0.04545 \times 22) = 0.04545\,(2 - 0.9999) = 0.04545\,(1.00002) = 0.0454509$

∴ The reciprocal of 22 is 0.0454509.

(c) Let $f(x) = x^m - \mathrm{N} = 0$, where N is the number whose n^{th} root is to be founded.

The solution to $f(x)$ is then $x = \mathrm{N}^{1/m}$. Here $f'(x) = mx^{m-1}$

By Newton-Raphson method, $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{(x_i^m - \mathrm{N})}{mx_i^{m-1}}$

$$= \frac{mx_i^m - x_i^m + \mathrm{N}}{mx_i^{m-1}} = \frac{(m-1)x_i^m + \mathrm{N}}{mx^{m-1}} = \frac{1}{m}\left[(m-1)x_i + \frac{\mathrm{N}}{x_i^{m-1}}\right] = \frac{x_i}{m}\left[(m-1) + \frac{\mathrm{N}}{x_i^m}\right]$$

Hence the real root of the equation correct to 4 decimal places is – 2.09145.

Example 7 : Derive a formula to find the cube root of N using Newton Raphson method hence find the cube root of 15.

[JNTU (H) June 2011 (Set No. 1)]

Solution : Let $f(x) = x^3 - N = 0$, when N is the number whose cube root is to be found.

The solution to $f(x)$ is then $x = N^{1/3}$

Here $f'(x) = 3x^2$

Using Newton - Raphson formula,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 - N}{3x_i^2}$$

$$= \frac{3x_i^3 - x_i^3 + N}{3x_i^2} = \frac{2x_i^3 + N}{3x_i^2}$$

$$\therefore \; x_{i+1} = \frac{1}{3}\left(2x_i + \frac{N}{x_i^2}\right) \qquad \ldots (1)$$

[JNTU (A) Dec. 2015]

Using the above iteration formula, the cube root of any number can be found out.

To find the cube root of 15

Let $N = 15$

Let the initial approximation be $x_0 = 2.4$

Substituting in (1), we get

$$x_1 = \frac{1}{3}\left(5 + \frac{15}{(2.5)^2}\right) = \frac{1}{3}\left(5 + \frac{15}{6.25}\right)$$

$$= \frac{1}{3}\left(5 + \frac{3}{1.25}\right) = \frac{1}{3}\left(\frac{6.25+3}{1.25}\right) = \frac{1}{3}\left(\frac{9.25}{1.21}\right) = 2.4666$$

Put $i = 1$ in (1). Then

$$x_2 = \frac{1}{3}\left(2x_1 + \frac{15}{x_1^2}\right)$$

$$= \frac{1}{3}\left[4.932 + \frac{15}{(2.466)^2}\right] = \frac{1}{3}\left[4.932 + \frac{15}{6.08}\right] = \frac{1}{3}[4.932 + 2.467] = 2.465$$

Put $i = 2$ in (1). Then

$$x_3 = \frac{1}{3}\left(2x_2 + \frac{15}{x_2^2}\right) = \frac{1}{3}\left[2 \times 2.405 + \frac{15}{(2.465)^2}\right] = \frac{1}{3}\left[4.93 + \frac{15}{6.076}\right]$$

$$= \frac{1}{3}[4.93 + 2.468] = \frac{1}{3}[7.3987] = 2.4662$$

Put $i = 3$ in (1). Then

$$x_4 = \frac{1}{3}\left[2x_3 + \frac{15}{(x_3)^2}\right]$$

$$= \frac{1}{3}\left[2 \times 2.4662 + \frac{15}{(2.4662)^2}\right] = \frac{1}{3}\left[4.9324 + \frac{15}{6.0821}\right] = 2.4661$$

The value is converging to 2.466. We take $\sqrt[3]{15} = 2.466$.

Example 11 : Find a real root for $x \tan x + 1 = 0$ using Newton Raphson method
[JNTU (K) Dec. 2015 (Set

(or) Find the root of the equation $x \sin x + \cos x = 0$ using Newton Raphson method.
[JNTU Sep 2006, (A) June 2011, (K) Dec. 2015 (Set

Solution : Given $f(x) = x \tan x + 1 = 0$. $\therefore$ $f'(x) = x \sec^2 x + \tan x$

Now $f(2) = 2 \tan 2 + 1 = -3.370079 < 0$ and $f(3) = 3 \tan 3 + 1 = .572370 >$

$\therefore$ The root lies between 2 and 3. We take the average of 2 and 3.

Let $x_0 = \dfrac{2+3}{2} = 2.5$

Using Newton-Raphson method, $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$, we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{-.86755}{3.14808} = 2.77558$$

$\rightarrow x\,(1/\cos x)^2 + \tan x$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.77558 - \frac{(-.06383)}{2.80004} = 2.798$$

$$\text{and } x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.798 - \frac{-0.0010803052}{2.7983} = 2.798$$

Since $x_2 = x_3$, therefore, the real root is 2.798.

Example 12 : Find a root of $e^x \sin x = 1$ (near 1) using Newton-Raphson's met
[JNTU Sep. 2006, (H) June 2010 (Set

Solution : Given $e^x \sin x = 1$

Let $f(x) = e^x \sin x - 1 \Rightarrow f'(x) = e^x(\sin x + \cos x)$

We have to find x_1 and x_2 such that $f(x_1)$ and $f(x_2)$ have opposite signs. Then the lies between x_1 and x_2.

$\therefore$ Root of the equation lies between x_1 and x_2.

$$f(0) = e^0 \sin 0 - 1 = -1 < 0; \; f(1) = e^1 \sin 1 - 1 = 1.287 > 0$$

$\therefore$ Root of the equation lies between 0 and 1.

By Newton-Raphson's method, $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Let $x_0 = \dfrac{1+0}{2} = 0.5$. Then $f(x_0) = e^{0.5}\sin(.5) - 1 = -0.20956$ and

$f'(x_0) = e^{0.5}[(\sin(.5) + \cos(.5)] = 2.237328$

$$\therefore\ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{-.20956}{2.237328} = 0.593665.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.593665 - \frac{e^{.593665}\sin(.593665) - 1}{e^{.593665}(\sin(.593665) + \cos(.593665))}$$

$$= 0.593665 - \frac{.01286}{2.51367} = 0.58854$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.58854 - \frac{.000018127}{2.4983} = 0.58853$$

Since $x_2 = x_3 = 0.58853$, the desired root of the equation is .58853.

Example 13 : Find a real root of the equation $xe^x - \cos x = 0$ using Newton Raphson method. **[JNTU 2006S, (A) June 2009, (K) May 2016, Oct. 2018 (Set No. 3)]**

(or) Using Newton-Raphson's method, find a positive root of $\cos x - x\,e^x = 0$.

[JNTU Sep. 2008S (Set No.1)]

(or) Find two values of x between which the root of $xe^x = \cos x$ lies.

[JNTU (H) May 2016]

Solution : Given $xe^x - \cos x = 0$. Let $f(x) = xe^x - \cos x = 0$

We have to find x_1 and x_2 such that $f(x_1)$ and $f(x_2)$ are of opposite signs.

$\therefore$ Root of the equation lies between x_1 and x_2.

We have $f(x) = xe^x - \cos x \Rightarrow f'(x) = (x+1)e^x + \sin x$

Now $f(0) = 0 - \cos 0 = -1 < 0$; $f'(0) = 1 + \sin 0 = 1$

$f(1) = e - \cos 1 = 2.177979 > 0$; $f'(1) = 6.27803$

$\therefore$ Root lies between 0 and 1.

Let $x_0 = \dfrac{0+1}{2} = 0.5$

By Newton-Raphson method, we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad \dots (1)$$

Putting $i = 0$ in (1), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{(-0.053221926)}{2.952507} = 0.51803$$

Now $f(x_1) = (0.51803)e^{0.51803} - \cos(0.51803) = 0.00083$

and $f'(x_1) = (1.51803)\,e^{0.51803} + \sin(0.51803) = 3.0435$

Putting $i = 1$ in (1), we get

$$\therefore\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.51803 - \frac{0.00083}{3.0435} = 0.5178$$

Now $f(x_2) = (0.5178)\,e^{0.5178} - \cos(0.5178) = 0.00013$

and $f'(x_2) = 1.5178\,e^{0.5178} + \sin(0.5178) = 3.04234$

Putting $i = 2$ in (1), we get

$$\therefore\ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5178 - \frac{0.0003}{3.04234} = 0.5177$$

Now $f(x_3) = (0.5177)\,e^{0.5177} - \cos(0.5177) = -0.0001745$

and $f'(x_3) = (1.5177)\,e^{0.5177} + \sin(0.5177) = 3.04183$

Similarly,

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.5177 + \frac{0.0001745}{3.04183} = 0.5177573$$

Since $x_3 = x_4 = 0.5177$, the desired root of the equation is 0.5177.

Example 14 : Find a real root of $x + \log_{10} x - 2 = 0$ using Newton Raphson me

[JNTU April 2007, (K) Feb. 2015, April 2019 (Se

Solution : Let $y = x + \log_{10} x - 2$ ... (1)

We obtain a rough estimate of the root by drawing the graph of (1) with the help following table.

x	1	2	3	4
y	−1	0.3010	1.4771	2.6021

Since the curve crosses x-axis at $x_0 = 1.8$, we take it as the initial approximation root.

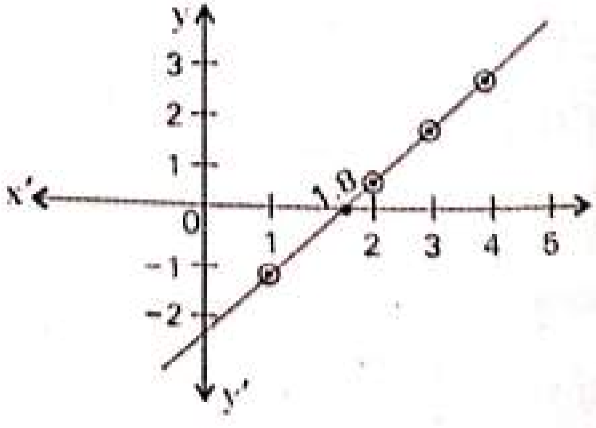

Fig. 6

We have $f(x) = x + \log_{10} x - 2 \Rightarrow f'(x) = 1 + \frac{1}{x}\log_{10} e$

$\therefore f(1.8) = 1.8 + \log_{10} 1.8 - 2 = 1.8 + 0.2552725 - 2 = 0.0555272$

and $f'(1.8) = 1 + \frac{1}{1.8}\log_{10} e = 1 + \frac{0.4343}{1.8} = 1.2412778$

By Newton - Raphson method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.8 - \frac{f(1.8)}{f'(1.8)} = 1.8 - \frac{0.0555272}{1.2412778} = 1.7552661$$

Now $f(x_1) = -0.00013658$; $f'(x_1) = 1.247369$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.755470 + \frac{0.00013658}{1.247369} = 1.75558$$

Now $f(x_2) = -0.0000001238$, $f'(x_2) = 1.2473536$.

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.75558 + \frac{0.0000001238}{1.2473536} = 1.75558$$

Since $x_2 = x_3$, the real root of $x + \log_{10} x - 2 = 0$ is 1.75558.

Example 15 : Using Newton-Raphson method, find a positive root of $x^4 - x - 9 = 0$.
[JNTU (A) June 2009, (K) May 2016 (Set No. 2)]

Solution : Let $f(x) = x^4 - x - 9$

Now $f(0) = -9 < 0$, $f(1) = -9 < 0$, $f(2) = 5 > 0$

$\therefore$ The root lies between 1 and 2.

Now $f(1.5) = -5.4375$, $f(1.75) = -1.3711$, $f(1.8) = 0.3024$, $f(1.9) = 2.1321$, $f(2) = 5$

Since $f(1.75) < 0$ and $f(1.8) > 0$, the root lies between 1.75 and 1.8.

Now $f'(x) = 4x^3 - 1$

$\therefore f'(1.8) = 4(1.8)^3 - 1 = 22.328$

By Newton-Raphson method, $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

Since $f(x)$ and $f'(x)$ have same sign at 1.8, we choose 1.8 as starting point.

i.e., $x_0 = 1.8$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.8 - \frac{f(1.8)}{f'(1.8)} = 1.8 - \frac{0.3024}{22.328} = 1.8 - 0.0135 = 1.7865$$

Now $f(x_1) = f(1.7865) = (1.7865)^4 - 1.7865 - 9 = -0.6003 < 0$

and $f'(x_1) = 4(1.7865)^3 - 1 = 21.807$

$$\therefore\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.7865 + \frac{0.6003}{21.807} = 1.814$$

Now $f(x_2) = (1.814)^4 - 1.814 - 9 = 0.014$

and $f'(x_2) = 4\,(1.814)^3 - 1 = 22.8766$

$$\therefore\ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.814 - \frac{0.014}{22.8766} = 1.8134$$

Now $f(x_3) = (1.8134)^4 - 1.8134 - 9 = 0.000303$

and $f'(x_3) = 4\,(1.8134)^3 - 1 = 22.8529$

$$\therefore\ x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.8134 - \frac{0.000303}{22.8529} = 1.8134$$

Since $x_3 = x_4 = 1.8134$, the desired root is 1.8134.

Note : Using Newton - Raphson method find an approximate root, which lies near $x=$ the equation $x^4 - x - 9 = 0$ upto two approximations. **[JNTU (K) Oct. 2018 (Set No.**

Example 16 : Find a real root of $x^3 - x - 2 = 0$, using Newton-Raphson method

[JNTU (A) June 2009 (Se

Solution : Let $f(x) = x^3 - x - 2$. Then $f'(x) = 3x^2 - 1$

Since $f(1) = 1 - 1 - 2 = -2$, $f(2) = 8 - 2 - 2 = 4$, therefore one root lies between 1

By Newton – Raphson method, $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ ... (1)

We take $x_0 = 1$. Then we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-2}{2} = 2 \quad \text{(Putting } i = 0 \text{ in (1))}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{11} = 1.6364 \quad \text{(Putting } i = 1 \text{ in (1))}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.6364 - \frac{f(1.6364)}{f'(1.6364)} \quad \text{(Putting } i = 2 \text{ in (1))}$$

$$= 1.6364 - \frac{0.7455}{7.0334} = 1.6364 - 0.106 = 1.5304$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.5304 - \frac{0.054}{6.02637} = 1.52144 \quad \text{(Putting } i = 3 \text{ in (1))}$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.52144 - \frac{0.0003584}{5.94434} = 1.5214 \quad \text{(Putting } i = 4 \text{ in (1))}$$

Since $x_4 = x_5$, the desired root is 1.5214.

Example 17 : By using Newton-Raphson method, find the root of $x^4 - x - 10 = 0$, correct to three places of decimal.

Solution : Let $f(x) = x^4 - x - 10$. We have $f(1) = -10 < 0$ and $f(2) = 4 > 0$

So there is a real root of $f(x) = 0$ lying between 1 and 2.

Now $f'(x) = 4x^3 - 1$. Here we take $x_0 = 2$

$x_0 = 2, f(x_0) = 4, f'(x_0) = 3$

The first approximation of the root is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2 - \dfrac{4}{31} = \dfrac{58}{31} = 1.871$

Second Aproximation :

$f(x_1) = 0.3835, \ f'(x_1) = 25.1988$

$$\therefore \quad x_2 = 1.871 - \frac{0.3835}{25.1988} = 1.85578$$

Third Approximation :

$f(x_2) = 0.004827, \ f'(x_2) = 24.5646$

$$\therefore \quad x_3 = 1.85578 - \frac{0.004827}{24.5646} = 1.85558$$

Hence the root is 1.856 corrected to three decimal places.

Example 18 : Find a real root of the equation $\cos x - x^2 - x = 0$ using Newton Raphson method. **[JNTU (H) Jan. 2012 (Set No. 1)]**

Solution : Given equation is $f(x) = \cos x - x^2 - x = 0 \Rightarrow f'(x) = -\sin x - 2x - 1$

Now $f(0) = 1, f(1) = \cos(1) - 1 - 1 < 0$. Thus the root lies between 0 and 1.

We will use the formula, $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ by Newton - Raphson method.

Take $x_0 = 0$, $f'(x_0) = f'(0) = -1$

$\therefore$ The first approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(1)}{(-1)} = 1$$

Now $f(x_1) = f(1) = \cos(1) - 1 - 1 = 0.5403 - 2 = -1.4597$

and $f'(x_1) = -\sin(1) - 2 - 1 = -3 - 0.8414 = -3.8414$

$\therefore$ The second approximation is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{(-1.4597)}{-3.8414} = 1 - 0.3799 = 0.6201$$

Now $f(x_2) = \cos(0.6201) - (0.6201)^2 - (0.6201) = 0.8138 - (0.3845) - (0.6201)$

$= -(0.1908)$

and $f'(x_2) = -\sin(0.6201) - 2(0.6201) - 1$

$= -(0.5811) - (1.2402) - 1 = -2.8213$

$\therefore\ x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 0.6201 - \dfrac{(-0.1908)}{(-2.8213)}$

$= 0.6201 - 0.0676 = 0.5525$

Now $f(x_3) = 0.8512 - 0.3052 - (0.5525) = -0.0065$

and $f'(x_3) = -(0.5248) - 1.105 - 1 = -2.6298$

$\therefore\ x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)} = 0.5525 - \dfrac{(-0.0065)}{-2.6298}$

$= 0.5525 - 0.0024 = 0.5501$

Now $f(x_4) = 0.8524 - 0.8527 = -0.0003$

Since $x_3 \approx x_4$, the approximate value of the root is 0.55.

Example 19 : Find a real root of the equation $3x - \cos x - 1 = 0$ (or $3x = \cos x + 1$) Newton Raphson method. **[JNTU (H) June 2012, (K) May 2016** (Set \

Solution : Let $f(x) = 3x - \cos x - 1$

$f(0) = 0 - \cos 0 - 1 = -2 < 0$

$f(1) = 3 - \cos 1 - 1 = 1 - 0.5403 = 0.4597 > 0$

$\therefore$ The root lies between 0 and 1.

$f'(x) = 3 + \sin x$

$\Rightarrow f'(1) = 3 + \sin(1) = 3 + 0.8414 = 3.8414$

By Newton-Raphson method,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{.... (1)}$$

Taking $i = 0$ and $x_0 = 1$ in (1), we get

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 1 - \dfrac{0.4597}{3.8414}$

$= 1 - 0.1196 = 0.8804$

$\therefore\ f(x_1) = f(0.8804) = 2.6412 - 0.6368 - 1 = 1.0044$ and $f'(x_1) = 3.7709$

From (1), $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = (0.8804) - 0.2663 = 0.6141$

$\therefore\ f(x_2) = 1.8423 - 0.8172 - 1 = 0.0251$ and $f'(x_2) = 3.5762$

From (1), $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 0.6141 - \frac{0.0251}{3.5762} = 0.6141 - 0.0070 = 0.6071$$

$\therefore f(x_3) = 1.8213 - 0.8213 - 1 = 0$

Hence the root of the equation is 0.6071

Example 20 : Find a real root of $x\log_{10} x = 1.2$ using Newton-Raphson method.

[JNTU (K) Feb. 2014 (Set No. 2)]

(or) Write the two approximations of $x\log_{10} x = 1.2$ by Newton-Raphson method.

[JNTU (K) Nov. 2018]

Solution : Let $f(x) = x\log_{10} x - 1.2$

We have, $f(2) = (2)\log(2) - 1.2 = 0.6020 - 1.2 = -0.5979 < 0$

and $f(3) = 3\log(3) - 1.2 = 1.4313 - 1.2 = 1.2313 > 0$

$\therefore$ The root lies between 2 and 3. Let $x_0 = \frac{2+3}{2} = 2.5$

By Newton-Raphson Method, $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

We have $f(x) = x\log_{10} x - 1.2$

$f'(x) = \log_{10} x + \log_{10} e = \log_{10}(ex)$, where $e = 2.7182$

The first approximation of the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.5 - \frac{f(2.5)}{f'(2.5)} = 2.5 - \left[\frac{(2.5)\log_{10} 2.5 - 1.2}{\log_{10}(2.5e)}\right]$$

$$= 2.5 + \left[\frac{0.2052}{0.8322}\right] = 2.5 + 0.2565 = 2.7435$$

The second approximation of the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7435 - \left[\frac{(2.7435)\log(2.7435) - 1.2}{\log(2.7435 \times 2.7182)}\right]$$

$$= 2.7435 - \left[\frac{1.2024 - 1.2}{\log(7.4573)}\right]$$

$$= 2.7435 - \frac{0.0024}{0.8725} = 2.7435 - 0.0027 = 2.7408$$

The third approximation of the root is

$$x_3 = 2.7408 - \left[\frac{2.7408 \log(2.7408) - 1.20}{\log(2.7408 + 2.7182)}\right]$$

$$= 2.7082 - \left[\frac{1.2001 - 1.2}{\log(7.4500)}\right]$$

$$= 2.7462 - \frac{0.0001}{0.8724} = 2.7462 - 0.00001 \;\; = 2.7461$$

We take this is the approximate vlaue of the root.

4.10 NEWTON-RAPHSON METHOD FOR SOLUTION OF SIMULTANEOUS EQUATI

Let the non-linear simultaneous equations be given by $f(x, y) = 0$ and $g(x, y) = 0$ whose real roots are required with in a specified accuracy.

Let (x_0, y_0) be an initial approximation to the root of the system (1).

If $(x_0 + h, y_0 + k)$ is the root of the system, then we must have

$$\left.\begin{aligned} f(x_0 + h, y_0 + k) = 0 \\ g(x_0 + h, y_0 + k) = 0 \end{aligned}\right\} \quad \ldots (2)$$

Assuming that f and g are sufficiently differentiable, we expand (2) by Taylor's we obtain

$$\left.\begin{aligned} f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} + \ldots\ldots = 0 \\ g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} + \ldots\ldots = 0 \end{aligned}\right\} \quad \ldots (3)$$

where $\frac{\partial f}{\partial x_0} = \left[\frac{\partial f}{\partial x}\right]_{x=x_0}$

and $f_0 = f(x_0, y_0) \ldots\ldots$

Neglecting the second and higher order terms, we obtain the following linear eq

$$\left.\begin{aligned} f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = 0 \\ g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = 0 \end{aligned}\right\} \quad \ldots (4)$$

Equation (3) possesses a unique solution if the Jacobian

$$J(f, g) = \begin{vmatrix} \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial y_0} \\ \frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial y_0} \end{vmatrix} \text{ does not vanish.}$$

The solution is given by

$$\left.\begin{aligned} h=\frac{1}{J(f,g)}&=\begin{vmatrix}-f & \frac{\partial f}{\partial y}\\ -g & \frac{\partial g}{\partial y}\end{vmatrix}\\ k=\frac{1}{J(f,g)}&=\begin{vmatrix}\frac{\partial f}{\partial x} & -f\\ \frac{\partial g}{\partial x} & -g\end{vmatrix}\end{aligned}\right\} \quad \ldots (5)$$

The new approximations are then given by

$$x_1 = x_0 + h \text{ and } y_1 = y_0 + k \quad \ldots (6)$$

This process will be repeated till we obtain the roots to the desired accuracy. If the iteration converges, it does quadratically. We give below a theorem (without proof) which gives sufficient condition for convergence.

Theorem. Let (x_0, y_0) be an approximation to a root (α,β) of the system of equation $f(x,y)=0$ are $g(x,y)=0$ ... (1) in the closed neighbourhood R containing (α,β).

If (*i*) if f, g and all their first and second derivatives are continuous and bounded in R.

(*ii*) $J(f,g) \neq 0$ in R, then the sequence of approximations given by

$$x_{i+1} = x_i - \frac{1}{J(f,g)}\begin{vmatrix} f & g \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y}\end{vmatrix} \text{ and } y_{i+1} = y_i - \frac{1}{J(f,g)}\begin{vmatrix} g & f \\ \frac{\partial g}{\partial x} & \frac{\partial f}{\partial y}\end{vmatrix}$$

converges to the root (α,β) of system (1)

Example 5 : Solve the system of equations by Newton-Raphson met $3yx^2-10x+7=0$ and $y^2-5y+4=0$. [JNTU (K) Dec. 2016 (MM-Set N

Solution : We will solve the system of equation using Newton-Raphson method.

Given $f(x,y)=3yx^2-10x+7; g(x,y)=y^2-5y+4$ ... (1)

Take $x=\frac{7}{3}=2.33, y=1$ in $f(x,y)$ and $g(x,y)$

$$f_0=3(1)(2.33)^2-10(2.3)+7=16.2867-23+7=0.2867$$

$$g_0=0$$

$$\frac{\partial f}{\partial x}=6xy-10, \frac{\partial f}{\partial y}=3x^2, \frac{\partial g}{\partial x}=0, \frac{\partial g}{\partial y}=2y-5 \quad \text{... (2)}$$

$$\frac{\partial f}{\partial x_0}=6(2.33)(1)-10=13.98-10=3.98, \frac{\partial g}{\partial x_0}=0$$

$$\frac{\partial f}{\partial y_0}=3(2.33)^2=16.2867, \frac{\partial f}{\partial y_0}=-3$$

Consider $J(f,g)=\begin{vmatrix} 3.98 & 16.2867 \\ 0 & -3 \end{vmatrix} \neq 0$

Thus the convergence criteria is satisfied.

We have $f_0+h\frac{\partial f}{\partial x_0}+k\frac{\partial f}{\partial y_0}=0$ ... (3)

$$\Rightarrow 0.2867+h(3.98)+k(16.2867)=0$$

$$g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = 0 \qquad \text{... (4)}$$

$\Rightarrow 0 + h(0) + k(-3) = 0 \Rightarrow k = 0$

From (3), $0.2867 + h(3.98) = 0$

$$\Rightarrow h = \frac{0.2867}{3.98} = 0.0720$$

First Approximation :

$x_1 = x_0 + h = 2.33 + 0.720 = 2.402$

$y_1 = y_0 + k = 1 + 0 = 1$

Second Approximation :

$f_1 = f_1(x_1, y_1) = 3(1)(2.402)^2 - 10(2.402) + 7$

$= 17.3088 - 24.02 + 7 = 24.3088 - 24.02 = 0.2888$

$g_1 = g_1(x_1, y_1) = y_1^2 - 5y_1 + y = 0$

$$\frac{\partial f}{\partial x_1} = 6(2.402)^2(1) - 10 = 34.6176 - 10 = 24.6176$$

$$\frac{\partial f}{\partial y_1} = 3(2.402)^2 = 17.3088$$

$$\frac{\partial g}{\partial x_1} = 0, \frac{\partial g}{\partial y_1} = 2 - 5 = -3$$

$$J(f,g) = \begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial y_1} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial y_1} \end{vmatrix} = \begin{vmatrix} 24.6176 & 17.3088 \\ 0 & -3 \end{vmatrix} \neq 0$$

Thus, the condition for convergence is satisfied.

We have, $f_1 + h\left(\frac{\partial f}{\partial x_1}\right) + k\frac{\partial f}{\partial y_1} = 0$ ... (5)

$$g_1 + h\left(\frac{\partial g}{\partial x_1}\right) + k\frac{\partial g}{\partial y_1} = 0 \qquad \text{... (6)}$$

Substituting the values in (5), (6)

$0.2888 + h(24.6176) + k(17.3088) = 0$ and $0 + h(0) + k(-3) = 0$

$\Rightarrow k = 0$

$h(24.6176) = -0.2888$

$h = -0.0117$

We have, $x_2 = x_1 + h = 2.402 - 0.0117 = 2.3903$

$y_2 = y_1 + k = y_1 + 0 = 1$

We take $x = 2.3903$ and $y = 1$ as approximate values of roots of the equations

Example 6 : Solve the system of equations by Newton Raphson method $x^2+y^2-1=0$ and $y-x^2=0$ [JNTU (K) Dec. 2016 (MM-Set No. 2)]

Solution : We will solve the given system of equation using Newton-Raphson method.

Given equations are $x^2+y^2-1=0; y-x^2=0$ ... (1)

Take, $f(x,y)=x^2+y^2-1, g(x,y)=y-x^2$

and $x=1.11, y=1.23$

$f_0=f(1.11,1.23)=1.2321+1.5178-1=1.7499$

$g_0=g(1.1,1.23)=0.02$

$\frac{\partial f}{\partial x}=2x, \frac{\partial f}{\partial y}=2y; \quad \frac{\partial g}{\partial x}=-2x, \frac{\partial g}{\partial y}=1$ .... (2)

$\frac{\partial f}{\partial x_0}=2(1.11)=2.22$

$\frac{\partial f}{\partial y_0}=2(1.23)=2.46$

$\frac{\partial g}{\partial x_0}=-2(1.11)=-2.22$

$\frac{\partial g}{\partial y_0}=1$

$J(f,g)=\begin{vmatrix} 2.22 & 2.46 \\ -2.22 & 1 \end{vmatrix} \neq 0$

∴ The condition for convergence is satisfied.

We take

$f_0+h\left(\frac{\partial f}{\partial x}\right)+k\left(\frac{\partial f}{\partial y_0}\right)=0, g+h\left(\frac{\partial g}{\partial x_0}\right)+k\left(\frac{\partial g}{\partial y_0}\right)=0$ ... (3)

$\Rightarrow 1.7499+h(2.22)+k(2.46)=0$

$0.02+h(-2.22)+k(1)=0$

adding, $1.7699+k(3.46)=0$

$k=\frac{-1.7694}{3.46}=-0.5115$

Substituting the values of k,

$1.7499+h(2.22)-(0.5115)(2.46)=0$

$\Rightarrow 1.7499+h(2.22)-1.2583=0$

$\Rightarrow h=-(0.2214)$

First Approximation

$x_1 = x_0 + h = 1.11 - 0.2214 = 0.8886$

$y_1 = y_1 + k = 1.23 - 0.5115 = 0.7185$

Second Approximation

$f_1 = (0.8886)^2 + (0.7185)^2 - 1$

$= 0.7896 + 0.5162 - 1 = 0.3058$

$g_1 = -0.0711$

$\frac{\partial f}{\partial x_1} = 2(0.8886) = 1.7772$

$\frac{\partial f}{\partial y_1} = 2(0.7185) = 1.4370$

$\frac{\partial g}{\partial x_1} = -1.7772, \frac{\partial g}{\partial y_1} = 1$

$J(f,g) = \begin{vmatrix} 1.7772 & 1.4370 \\ -1.7772 & 1 \end{vmatrix} \neq 0$.

The condition for convergence is satisfied.

$f_1 + h\left(\frac{\partial f}{\partial x_1}\right) + k\left(\frac{\partial f}{\partial y_1}\right) = 0; \quad g_1 + h\left(\frac{\partial g}{\partial y_1}\right) + k\left(\frac{\partial g}{\partial y_1}\right) = 0$ ... (4)

$0.3058 + h(1.7772) + k(1.4370) = 0$

$-0.0711 + h(-1.7772) + k(1) = 0$

adding, $0.2347 + k(2.4370) = 0$

$\Rightarrow k = -0.0963$

Substituting the value of k

$-0.0711 - 0.0963 = h(1.7772)$

$-0.1674 = h(1.7772)$

$h = -(0.0941)$

$x_2 = x_1 + h = 0.8886 - 0.0941 = 0.7945$

$y_2 = y_1 + k = 0.7185 - 0.0963 = 0.6222$

These values are taken as the approximate roots of the equation.

CHAPTER **5** **Interpolation**

5.1 INTRODUCTION

If we consider the statement $y = f(x)$, $x_0 \le x \le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function $f(x)$ is single valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, ..., x_n$ can be calculated. The problem now is if we are given the set of tabular values

x	x_0	x_1	x_2	...	x_n
y	y_0	y_1	y_2	...	y_n

satisfying the relation $y = f(x)$ and the explicit definition of $f(x)$ is not known, is it possible to find a simple function say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process of finding $\phi(x)$ is called **interpolation**. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation.

5.2 ERRORS IN POLYNOMIAL INTERPOLATION

Suppose the function $y(x)$ which is defined at the points $(x_i, y_i), i = 0,1,2,3,...,n$ is continuous and differentiable $(n+1)$ times. Let $\phi_n(x)$ be polynomial of degree not exceeding n such that $\phi_n(x_i) = y_i, i = 0,1,2,3,...,n$ ...(1)

be the approximation of $y(x)$ using this $\phi_n(x_i)$ for other value of x, not defined by (1). The error is to be determined. Since $y(x) - \phi_n(x) = 0$ for $x = x_0, x_1, x_2, ..., x_n$ we put

$$y(x) - \phi_n(x) = \text{L}\,\pi_{n+1}(x) \quad ...(2)$$

where $\pi_{n+1}(x) = (x - x_0)(x - x_1) .... (x - x_n)$ ...(3)

and L to be determined such that the equation (2) holds for any intermediate value of x such as $x = x', x_0 < x' < x_n$.

$$\text{Clearly L} = \frac{y(x') - \phi_n(x')}{\pi_{n+1}(x')} \quad ...(4)$$

We construct a function F(x) such that F$(x) = y(x) - \phi_n(x) - \pi_{n+1}(x)$ ...(5)

where L is given by (4).

We can easily see that F$(x_0) = 0 =$ F$(x_1) =$ F$(x_n) =$ F(x'). Then F(x) vanishes $(n+2)$ times in the interval $[x_0, x_n]$. Then by repeated application of Rolle's theorem F'(x) must be equal to zero $(n+1)$ times, F''(x) must be zero n times ... in the interval

$[x_0, x_n]$. Also $F^{n+1}(x) = 0$ once in this interval. Suppose this point is $x = \xi, x_0 < \xi$, Differentiate (5), $(n+1)$ times with respect to x and putting $x = \xi$, we get

$$y^{n+1}(\xi) - L\,\underline{|(n+1)} = 0 \text{ which implies that } L = \frac{y^{n+1}(\xi)}{\underline{|n+1}} \quad ...(6)$$

Comparing (4) and (6), we get, $y(x') - \phi_n(x') = \frac{y^{n+1}(\xi)}{\underline{|n+1}} \pi_{n+1}(x')$

which can be written as $y(x) - \phi_n(x) = \frac{\pi_{n+1}(x)}{\underline{|n+1}} y^{n+1}(\xi),\ x_0 < \xi < x_n$ ...(7)

This gives the required expression for error.

5.3 FINITE DIFFERENCES

1. Introduction :

In this chapter, we introduce what are called the forward, backward and ce differences of a function $y = f(x)$. These differences are three standard exampl finite differences and play a fundamental role in the study of Differential calculus, whi an essential part of Numerical Applied Mathematics.

2. Forward Differences :

Consider a function $y = f(x)$ of an independent variable x. Let $y_0, y_1, y_2, ...,$ the values of y corresponding to the values $x_0, x_1, x_2, ..., x_r$ of x respectively. Ther differences $y_1 - y_0, y_2 - y_1, ...$ are called the first forward differences of y, and we de them by $\Delta y_0, \Delta y_1, ...$. That is $\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2, ...$

In general, $\Delta y_r = y_{r+1} - y_r,\ r = 0, 1, 2, ...$ ...(1

Here the symbol Δ is an operator called the **Forward difference** operator.

The first forward differences of the first forward differences are called second for differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, ...$.

That is, $\Delta^2 y_0 = \Delta y_1 - \Delta y_0,\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1, ...$

In general, $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r, r = 0,1,2,...$ ...(2

Here Δ^2 is an operator called, **Second order Forward difference** operator.

Similarly, the n[th] forward differences are defined by the formula

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r, r = 0,1,2,... \quad ...(3$$

While using this formula for $n = 1$, use the notation $\Delta^0 y_r = y_r$.

If $f(x)$ is a constant function, i.e., if $f(x) = c$, a constant, then $y_0 = y_1 = y_2 = ...$ and we have $\Delta^n y_r = 0$ for $n = 1, 2, 3, ...$ and $r = 0, 1, 2, ...$. The symbol Δ^n is referr the n[th] forward difference operator.,

Note: $\Delta f(x) = f(x+h) - f(x)$

Properties : The first forward difference operator Δ have the following properties :

(i) $\Delta c = 0$ (differences of a constant function are zero)

(ii) $\Delta(cy(x)) = c(\Delta y(x))$, where c is a constant

(iii) $\Delta(u(x)+v(x)) = \Delta u(x) + \Delta v(x)$

(iv) $\Delta[u(x)v(x)] = u(x)\Delta v(x) + v(x+1)\Delta u(x)$ **(or)** $\Delta[f(x)g(x)] = f(x+h)\Delta g(x) + g(x)\Delta f(x)$

(v) $\Delta\left[\frac{u(x)}{v(x)}\right] = \frac{v(x)\Delta u(x) - u(x)\Delta v(x)}{v(x)v(x+1)}$ **(or)** $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}$

Note : Proofs for (iv) and (v) are given in Solved Example 1.

3. Forward Difference Table :

The forward differences are usually arranged in tabular columns as shown in the following table called a Forward Difference Table.

Values of x	Values of y	First differences	Second differences	Third differences	Fourth differences
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
--	--	Δy_3	---	---	---
x_n	y_n	---	---	---	---

Example : Finite Forward Difference Table for the function $y = x^3$

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

4. Backward Differences :

As mentioned earlier, let $y_0, y_1, y_2, ..., y_r, ...$ be the values of a function $y = j$ corresponding to the values $x_0, x_1, x_2, ..., x_r, ...$ of x respectively. Then

$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, ...$ are called the first back differences.

In general, $\nabla y_r = y_r - y_{r-1}, r = 1, 2, 3, ...$ ...(1

The symbol ∇ is called the **Backward difference** operator. Like the operator Δ, operator is also a Linear Operator.

Comparing expression (1) above with the expression (1) of previous section immediately note that $\nabla y_r = \Delta y_{r-1}, r = 0, 1, 2, ...$ ...(2

The first backward differences of the first backward differences are called se backward differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3, ..., \nabla^2 y_r, ...$ *i.e.,*

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2, ...$

In general, $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, \; r = 2, 3, ...$ ...(3

Similarly, the n[th] backward differences are defined by the formula

$$\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, \; r = n, \; n+1, ... \quad ...(4$$

While using this formula, for $n = 1$ we employ the notation $\nabla^0 y_r = y_r$.

If $y = f(x)$ is a constant function, then $y = c$, a constant, for all x, and w $\nabla^n y_r = 0$ for all n.

The symbol ∇^n is referred to as the n[th] backward difference operator.

Note: $\nabla f(x) = f(x) - f(x-h)$

5. Backward Difference Table :

The backward differences can be exhibited as shown in the following table, called the Backward Difference Table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\nabla^3 y_3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3		
x_3	y_3			

6. Central Differences :

With $y_0, y_1, y_2, \ldots, y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2, \ldots, x_r, \ldots$ of x, we define the first Central differences $\delta y_{1/2}$, $\delta y_{3/2}$, $\delta y_{5/2}, \ldots$ as follows :

$$\delta y_{1/2} = y_1 - y_0,\ \delta y_{3/2} = y_2 - y_1,\ \delta y_{5/2} = y_3 - y_2, \ldots,\ \delta y_{r-1/2} = y_r - y_{r-1} \quad \ldots(1)$$

The symbol δ is called the **Central difference** operator. This operator is a Linear operator.

Comparing expressions (1) above with expressions earlier used on Forward and Backward differences, we get

$$\delta y_{1/2} = \Delta y_0 = \nabla y_1,\ \delta y_{3/2} = \Delta y_1 = \nabla y_2,\ \delta y_{5/2} = \Delta y_2 = \nabla y_3, \ldots$$

In general, $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}$, $n = 0, 1, 2, \ldots$ ...(2)

The first central differences of the first central differences are called the second central differences and are denoted by $\delta^2 y_1$, $\delta^2 y_2$, $\delta^2 y_3$, ... Thus,

$$\delta^2 y_1 = \delta_{3/2} - \delta_{1/2},\ \delta^2 y_2 = \delta_{5/2} - \delta_{3/2}, \ldots\ \delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \quad \ldots(3)$$

Higher order Central differences are similarly defined. In general the n[th] central differences are given by :

(*i*) for odd n : $\delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}$, $r = 1, 2, \ldots$ ...(4)

(*ii*) for even n : $\delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2}$, $r = 1, 2, \ldots$ ...(5)

While employing the formula (4) for $n = 1$, we use the notation $\delta^0 y_r = y_r$.

If y is a constant function, that is, if $y = c$, a constant, then $\delta^n y_r = 0$, for all $n \geq 1$.

The symbol δ^n is referred to as the n[th] central difference operator.

7. Central Difference Table :

The central differences can be displayed in a table as shown below. This is calle Central difference Table.

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

5.4 DIFFERENCE BETWEEN INTERPOLATION AND EXTRAPOLATION

[JNTU (K) Dec. 2016 (Set No.

Interpolation is the process of computing the value of a function y for any intermedi value of the independent variable (x) while the process of finding the value of the functi for some value of x outside the given range is called extrapolation.

Interpolation : Interpolation is the estimation of an unknown quantity between known quantities or drawing conclusions about missing information from the availab information. Interpolation is useful where the data surrounding the missing data is availab and its trend and longer term cycles are known.

Extrapolation : Extrapolation is the statistical technique of inferring unknown fr the known data. This is used to predict future data by relying on historical data. This is u when the present circumstances do not indicate any interpolation in the long established p trends. Extrapolation is the process of finding a value outside a data set. This tool is use not only in statistics but also in science and business.

5.5 APPLICATIONS OF INTERPOLATION

[JNTU (K) Dec. 2016 (Set No.

1. Using interpolation we can take a set of data points $(x_i, g(x_i)) = (x_i, y_i)$ and tabula them into a continuous function. Such process is called curve fitting. The continuo function (curve) may characterise the relation between the variables x and continuously rather than discretely.
2. Using interpolation methods we can construct polynomials, splines and trigonome polynomials.

SOLVED EXAMPLES

Example 1 : The following table gives a set of values of x and the corresponding values of $y = f(x)$.

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

Form the forward difference table and write down the values of $\Delta f(10), \Delta^2 f(10), \Delta^3 f(15)$ and $\Delta^4 f(15)$.

Solution : The forward difference table for the given values of x and y is as shown below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	19.97					
		1.54				
15	21.51		−0.58			
		0.96		.67		
20	22.47		0.09		−0.68	
		1.05		−0.01		0.72
25	23.52		0.08		+0.04	
		1.13		0.03		
30	24.65		0.11			
		1.24				
35	25.89					

We note that the values of x are equally spaced with step-length $h = 5$.

$\therefore \quad x_0 = 10, x_1 = 15, ..., x_5 = 35$

and $\quad y_0 = f(x_0) = 19.97$

$y_1 = f(x_1) = 21.51$

$y_5 = f(x_5) = 25.89$

From table, we have

$\Delta f(10) = \Delta y_0 = 1.54; \quad \Delta^2 f(10) = \Delta^2 y_0 = -0.58;$

$\Delta^3 f(15) = \Delta^3 y_1 = -0.01; \quad \Delta^4 f(15) = \Delta^4 y_1 = 0.04$

Example 2 : Construct a forward difference table from the following data

x	0	1	2	3	4
y_x	1	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 y_1, y_5$ and y_5.

Solution : The forward difference table for the given values of x and y is as shown be

x	y_x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 0$	$y_0 = 1$				
		$\Delta y_0 = 0.5$			
$x_1 = 1$	$y_1 = 1.5$		$\Delta^2 y_0 = 0.2$		
		$\Delta y_1 = 0.7$		$\Delta^3 y_0 = 0$	
$x_2 = 2$	$y_2 = 2.2$		$\Delta^2 y_1 = 0.2$		$\Delta^4 y_0 = 0.4$
		$\Delta y_2 = 0.9$		$\Delta^3 y_1 = 0.4$	
$x_3 = 3$	$y_3 = 3.1$		$\Delta^2 y_2 = 0.6$		
		$\Delta y_3 = 1.5$			
$x_4 = 4$	$y_4 = 4.6$				

Now, $\Delta^3 y_1 = y_4 - 3y_3 + 3y_2 - y_1 = 4.6 - 3\,(3.1) + 3\,(2.2) - 1.5 = 0.4$

Again, we have

$$y_x = y_0 + {}^xC_1\Delta y_0 + {}^xC_2\Delta^2 y_0 + {}^xC_3\Delta^3 y_0 + {}^xC_4\Delta^4 y_0$$

$$= 1 + x\,(0.5) + \frac{1}{2!}\,(x\,(x-1))\,(0.2) + \frac{1}{3!}x\,(x-1)\,(x-2)\,(0)$$

$$+ \frac{1}{4!}x\,(x-1)\,(x-2)\,(x-3)\,(0.4)$$

$$= 1 + \frac{1}{2}x + \frac{1}{10}(x^2 - x) + \frac{1}{60}(x^4 - 6x^3 + 11x^2 - 6x)$$

$$\therefore\ y_x = \frac{1}{60}(x^4 - 6x^3 + 17x^2 + 18x + 60)$$

$$\Rightarrow y_5 = \frac{1}{60}\left((5)^4 - 6\,(5)^3 + 17\,(5)^2 + 18\,(5) + 60\right) = 7.5$$

Example 3 : If $f(x) = x^3 + 5x - 7$, form a table of forward differences fo $x = -1, 0, 1, 2, 3, 4, 5$. Show that the third differences are constant.

Solution : Here $f(-1) = -1 - 5 - 7 = -13$.

$f(0) = 0 - 7 = -7$,

$f(1) = 1 + 5 - 7 = -1$,

$f(2) = 8 + 10 - 7 = 11$,

$f(3) = 27 + 15 - 7 = 35$,

$f(4) = 64 + 20 - 7 = 77,$

$f(5) = 125 + 25 - 7 = 143$

We form the difference table as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-13			
		6		
0	-7			
		6	0	
1	-1			
		12	6	6
2	11			
		24	12	6
3	35			
		42	18	6
4	77			
		66		
5	143		24	6

We note from the table that all the third forward differences are constant. This illustrates the result discussed in 2.5

Example 4 : Given $f(-2) = 12, f(-1) = 16, f(0) = 15, f(1) = 18, f(2) = 20$ form the Central difference table and write down the values of $\delta y_{-3/2}, \delta^2 y_0$ and $\delta^3 y_{1/2}$ by taking $x_0 = 0$.

Solution : The Central difference table is

x	$y = f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
-2	12				
		4			
			-5		
-1	16				
				9	
		-1			
					-14
			4		
0	15				
				-5	
		3			
			-1		
1	18				
		2			
2	20				

Since $x_0 = 0$ and $h = 1$, we have $y_{-r} = f(x_0 - rh) = f(-r)$

From the above table, $\delta y_{-3/2} = \delta f(-3/2) = 4,\ \delta^2 y_0 = 4,\ \delta^3 y_{1/2} = -5$.

Example 5 : Find the missing term in the following table

X	0	1	2	3	4
Y	1	3	9	–	81

[JNTU (K) Dec. 2016 (MM-Set No. 2)]

Solution : Consider $\Delta^4 y_0 = 0$ (we are given only 4 values)

$\Rightarrow \; y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Substitute given values, we get

$81 - 4y_3 + 54 - 12 + 1 = 0 \; \Rightarrow \; y_3 = 31$.

Note : From the given data we can conclude that the given function is y

e have to assume that y is a polynomial function, which is not so. Thus we

$= 3^3 = 27$.

Solution : Consider $\Delta^4 y_0 = 0$ (we are given only 4 values)

$\Rightarrow\ y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Substitute given values, we get

$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31$.

Note : From the given data we can conclude that the given function is $y = 3^x$. To f... we have to assume that y is a polynomial function, which is not so. Thus we are not g... $y = 3^3 = 27$.

Example 6 : Show that $\delta^2 y_5 = y_6 - 2y_5 + y_4$

Solution : We know the formulae for Central difference operator δ,

$$\delta y_{r-1/2} = y_r - y_{r-1} \quad \text{... (1)}$$

and $$\delta^2 y_n = \delta y_{\frac{n+1}{2}} - \delta y_{\frac{n-1}{2}} \quad \text{... (2)}$$

From (1) & (2), we have

$$\delta^2 y_5 = \delta y_{11/2} - \delta y_{9/2} \quad \text{... (3)}$$

$$\delta y_{11/2} = y_6 - y_5 \quad \text{... (4)}$$

$$\delta y_{9/2} = y_5 - y_4 \quad \text{... (5)}$$

Using (4) and (5) we get

$$\delta^2 y_5 = y_6 - 2y_5 + y_4$$

which is the required result.

Example 7 : Prove that $\delta^2 f(x) = (\Delta - \nabla) f(x) = \Delta\nabla f(x)$

Solution : (*i*) By using definitions of the operators δ, Δ and ∇, we get

$$\delta^2 f(x) = \delta\{\delta f(x)\}$$

$$= \delta\left\{f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)\right\}$$

$$= \delta f\left(x + \frac{h}{2}\right) - \delta f\left(x - \frac{h}{2}\right)$$

$$= \{f(x+h) - f(x)\} - \{f(x) - f(x-h)\}$$

$$= (\Delta - \nabla) f(x) \quad \text{... (1)}$$

(*ii*) $$\Delta\nabla f(x) = \Delta\{\nabla f(x)\} = \Delta\{f(x) - f(x-h)\}$$

$$= \Delta f(x) - \Delta f(x-h)$$

$$= \Delta f(x) - \{f(x) - f(x-h)\}$$

$$= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x) \quad \text{... (2)}$$

The required result follows from (1) and (2)

5.6 SYMBOLIC RELATIONS AND SEPARATION OF SYMBOLS

We will define more operators and symbols in addition to Δ, ∇ and δ already defined and establish difference formulae by Symbolic methods.

AVERAGE OPERATOR :

Def. The averaging operator μ is defined by the equation

$$\mu y_r = \frac{1}{2}(y_{r+1/2} + y_{r-1/2})$$

or $$\mu y_x = \frac{1}{2}\left[y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right]$$

Def. The shift operator E is defined by the equation $Ef(x) = f(x+h)$ (or) $Ey_x = y_{x+h}$ (or) $E y_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. In other words, E is the operation of increasing the argument x by h so that $Ef(x) = f(x+h)$.

A second operation with E gives $E^2 y_r = E(E y_r) = E(y_{r+1}) = y_{r+2}$ or $E^2 f(x) = f(x+2h)$, etc.

Generalising $E^n y_r = y_{r+n}$ or $E^n f(x) = f(x+nh)$

Def. Inverse operator E^{-1} is defined as $E^{-1}f(x) = E^{-1}f(x-h)$ or $E^{-1}y_x = y_{x-h}$ or $E^{-1}y_r = y_{r-1}$.

In general, $E^{-n}y_r = y_{r-n}$.

RELATIONSHIP BETWEEN Δ AND E

We have $\Delta y_0 = y_1 - y_0 = E y_0 - y_0 = (E-1)y_0$

$\Rightarrow \Delta \equiv E - 1$ or $E = 1 + \Delta$ ...(1)

Alternatively, we have

$\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E-1)y_x$

This shows that the operators Δ and E are connected by the symbolic relation $\Delta = E - 1$.

SOME MORE RELATIONS

$$\Delta^3 y_0 = (E-1)^3 y_0 = (E^3 - 3E^2 + 3E - 1)y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = (E-1)^4 y_0 = (E^2 - 2E + 1)^2 y_0 = (E^4 + 4E^2 + 1 - 4E^3 - 4E + 2E^2)y_0$$

$$= (E^4 - 4E^3 + 6E^2 - 4E + 1)y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

We can easily establish the following relations:

(*i*) $\nabla = 1 - E^{-1}$ **[JNTU (K) Dec. 2016, April 2019 (Set No. 2)]**

(*ii*) $\delta = E^{1/2} - E^{-1/2}$

(*iii*) $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$ (*iv*) $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

(*v*) $\mu^2 = 1 + \frac{1}{4}\delta^2$ **[JNTU (K) Dec. 2015 (Set No. 2)]**

Proof: (*i*) $\nabla y_r = y_r - y_{r-1} = y_r - E^{-1}y_r = (1 - E^{-1})y_r$

$\therefore \nabla = 1 - E^{-1}$ or $E = (1-\nabla)^{-1}$

(*ii*) We have $\delta y_r = y_{r+\frac{1}{2}} - y_{r-\frac{1}{2}}$ or $\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$

$$\therefore \delta y_x = E^{1/2}y_x - E^{-1/2}y_x$$

$$= (E^{1/2} - E^{-1/2})y_x$$

$$\therefore \delta = E^{1/2} - E^{-1/2}$$

(*iii*) $\mu y_r = \frac{1}{2}(y_{r+1/2} + y_{r-1/2})$

$$= \frac{1}{2}[E^{1/2}y_r + E^{-1/2}y_r] = \frac{1}{2}[E^{1/2} + E^{-1/2}]y_r$$

$$\therefore \mu = \frac{1}{2}[E^{1/2} + E^{-1/2}].$$

(*iv*) $E\nabla y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h} = y_{x+h} - y_x = \Delta y_x$

$\therefore E\nabla = \Delta$ ... (1)

Also $\nabla E y_x = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x$. $\therefore \nabla E = \Delta$ ... (2)

From (1) and (2), $\Delta = E\nabla = \nabla E$ ... (3)

Now $\delta E^{1/2} y_x = \delta y_{\left(x+\frac{h}{2}\right)} = y_{\left(x+\frac{h}{2}+\frac{h}{2}\right)} - y_{\left(x+\frac{h}{2}-\frac{h}{2}\right)} = y_{x+h} - y_x = \Delta y_x$

$\therefore \delta E^{1/2} = \Delta$ ... (4)

Hence $\Delta = E\nabla = \nabla E = \delta E^{1/2}$ [From (3) and (4)]

(*v*) We know that $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$ ... (1) and $\delta = E^{1/2} - E^{-1/2}$ ... (2)

$$\mu^2 = \frac{1}{4}[E^{1/2} + E^{-1/2}]^2 = \frac{1}{4}[E + E^{-1} + 2]$$

$$= \frac{1}{4}[(E^{1/2} - E^{-1/2})^2 + 4] = \frac{1}{4}(\delta^2 + 4), \text{ using (2)}$$

$$\therefore \mu^2 = \frac{1}{4}(\delta^2 + 4).$$

Def. The operator D is defined as $Dy(x) = \frac{d}{dx}(y(x))$.

Relation between the operators D and E [JNTU (K) Feb. 2015 (Set No. 4)]

Using Taylor's series we have, $y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \ldots$

This can be written in symbolic form $Ey_x = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots\right]y_x = e^{hD} \cdot y_x$

($\because$ The above series in brackets is the expansion of e^{hD})

$\therefore$ We obtain the relation $E \equiv e^{hD}$. ...(3)

Note : Using the relation (1), many identities can be obtained. This relation is used to separate the effect of E into powers of Δ. This method of separation is called the method of separation of symbols. Some examples are given.

RELATIONSHIP BETWEEN THE OPERATORS

	E	Δ	∇	δ	hD
E	E	$\Delta + 1$	$(1-\nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{1}{4}\delta^2\right)}$	e^{hD}
Δ	$E - 1$	Δ	$(1-\nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{1}{4}\delta^2\right)}$	$e^{hD} - 1$
∇	$1 - E^{-1}$	$1 - (1+\Delta)^{-1}$	∇	$-\frac{1}{2}\delta + \delta\sqrt{\left(1 + \frac{1}{4}\delta^2\right)}$	$1 - e^{hD}$
δ	$E^{1/2} - E^{-1/2}$	$\Delta(1+D)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	δ	$2\sin\left(\frac{hD}{2}\right)$
μ	$\frac{1}{2}(E^{1/2} + E^{-1/2})$	$\left(1 + \frac{1}{2}\Delta\right)(1+\Delta)^{1/2}$	$\left(1 - \frac{1}{2}\nabla\right)(1-\nabla)^{-1/2}$	$\sqrt{\left(1 + \frac{1}{4}\delta^2\right)}$	$\cosh\left(\frac{hD}{2}\right)$
hD	$\log E$	$\log(1+E)$	$\log(1-\Delta)^{-1}$	$2\sinh^{-1}\left(\frac{\delta}{2}\right)$	

Example 2 : Prove that $\nabla = 1 - E^{-1}$ **[JNTU (K) Dec. 2016 (Set No. 2)]**

Solution : We have $\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x = (1 - E^{-1}) y_x$

This shows that the operators ∇ and E^{-1} (inverse shift operator) are connected by the symbolic relation $\nabla = 1 - E^{-1}$ or $E = (1-\nabla)^{-1}$

Note : Prove that $\Delta = E - 1$

Sol. We have $\Delta y_x = y_{x+h} - y_x = E\, y_x - y_x = (E-1) y_x$

$\therefore \Delta = E - 1$

Example 3 : Find $\Delta(e^{ax} \log bx)$ **[JNTU (K) Feb. 2014, 2015 (Set No. 2)]**

Solution :
$$\Delta(e^{ax} \log bx) = e^{ax+h} \log(b(x+h)) - e^{ax} \log bx$$
$$= e^{ax}.e^{h}.\log(bx + bh) - e^{ax} \log bx$$
$$= e^{ax}\left[e^{h} \log(bx+bh) - \log bx\right]$$

Alternate Method : Apply the property

$\Delta[f(x)g(x)] = f(x+h).\Delta g(x) + g(x)\Delta f(x)$ (Refer Example 7)

Example 4 : Evaluate (*i*) $\Delta \cos x$ (*ii*) $\Delta \log f(x)$ **[JNTU (K) Feb. 2015 (Set No. 1)]**
(*iii*) $\Delta^2 \sin(px+q)$ **[JNTU (K) Feb. 2015 (Set No. 2)]** (*iv*) $\Delta \tan^{-1} x$
(*v*) $\Delta^n e^{ax+b}$. **[JNTU (K) May 2016 (Set No. 4)]**

Solution : Let h be the interval of differencing. Then

(*i*) $\Delta \cos x = \cos(x+h) - \cos x = -2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$, using $\cos C - \cos D$ formula

(*ii*) $\Delta \log f(x) = \log f(x+h) - \log f(x) = \log\left(\frac{f(x+h)}{f(x)}\right)\left[\because \log\frac{a}{b} = \log a - \log b\right]$

$$= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right] = \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

(*iii*) $\Delta \sin(px+q) = \sin[p(x+h)+q] - \sin(px+q)$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}\quad\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2 \sin(px+q) = 2\sin\frac{ph}{2}\Delta\left[\sin\left(px+q+\frac{1}{2}(\pi+ph)\right)\right]$$

$$= \left[2\sin\frac{ph}{2}\right]^2\sin\left(px+q+2\cdot\frac{1}{2}(\pi+ph)\right)$$

Note : If $h=\frac{1}{2}$, find $\Delta^2\sin(px+q)$ **[JNTU (K) Feb. 2015 (Set No. 1)]**

(*iv*) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$$= \tan^{-1}\left[\frac{x+h-x}{1+x(x+h)}\right]\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left[\frac{h}{1+x(x+h)}\right]$$

(*v*) $\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b} = e^{ax+b}.e^{ah} - e^{ax+b}$

$$= e^{(ax+b)}(e^{ah}-1) \quad \ldots (1)$$

$\Delta^2 e^{ax+b} = \Delta[\Delta(e^{ax+b})] = \Delta[(e^{ah}-1)(e^{ax+b})]$; using (1)

$= (e^{ah}-1)\Delta(e^{ax+b})$ [$\because$ $e^{ah}-1$ is a constant]

$= (e^{ah}-1)^2 e^{ax+b}$

Proceeding like this, we get $\Delta^n(e^{ax+b}) = (e^{ah}-1)^n e^{ax+b}$.

Example 7 : Find (*i*) $\Delta[f(x)\,g(x)]$ [JNTU(K) April 2019 (Set No. 3)]

(*ii*) $\Delta\left[\dfrac{f(x)}{g(x)}\right]$. [JNTU(K) April 2019 (Set No. 4)]

Solution : Let h be the interval of differencing.

(*i*) $\Delta[f(x)\,g(x)] = f(x+h)\,g(x+h) - f(x)\,g(x)$

$= f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)$

$= f(x+h)\,[g(x+h) - g(x)] + g(x)\,[f(x+h) - f(x)]$

$= f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x)$.

(*ii*) $$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} = \frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x)\,g(x+h)}$$

$$= \frac{f(x+h)\,g(x) - f(x)\,g(x) + f(x)\,g(x) - f(x)\,g(x+h)}{g(x)\,g(x+h)}$$

$$= \frac{g(x)\,[f(x+h) - f(x)] - f(x)\,[g(x+h) - g(x)]}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}.$$

Example 8 : If the interval of differencing is unity, prove that $\Delta f(x) = \dfrac{-\Delta f(}{f(x)f(}$

[JNTU(H) June 2010 (Set

Solution : We know that

$$\Delta\left(\frac{1}{f(x)}\right) = \frac{1}{f(x+h)} - \frac{1}{f(x)} = \frac{-[f(x+h) - f(x)]}{f(x)\,f(x+h)} = \frac{-\Delta f(x)}{f(x)\,f(x+h)}$$

Taking $h = 1$, we get

$$\Delta\left(\frac{1}{f(x)}\right) = \frac{-\Delta f(x)}{f(x)\,f(x+1)}$$

Hence the result.

Example 9 : Find the second difference of the polynomial $x^4 - 12x^3 + 42x^2 -$ with interval of differencing $h = 2$. [JNTU 2008S, (H) Dec. 2011 (Set

Solution : Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$.

First difference of $f(x)$ is given by $\Delta f(x)$

$$\therefore \Delta f(x) = f(x+h) - f(x)$$
$$= f(x+2) - f(x)$$
$$= (x+2)^4 - 12(x+2)^3 + 42(x+2)^2 - 30x(x+2) + 9 - 9x^4 + 12x^3 - 42x^2 + 30x - 9$$
$$= 8x^3 - 48x^2 + 56x + 28$$

Second difference $= \Delta^2 f(x) = \Delta[\Delta f(x)] = \Delta[8x^3 - 48x^2 + 56x + 28]$
$$= 8(x+2)^3 - 48(x+2)^2 + 56(x+2) + 28$$
$$= -8x^3 + 48x^2 - 56x - 28 = 48x^2 - 96x - 16.$$

Example 10: (*i*) Show that $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$

[JNTU 2003, (K) Feb. 2014, Oct. 2018 (Set No. 4)]

(*ii*) If $f(x) = e^{ax}$, show that $\Delta^n f(x) = (e^{ah} - 1)^n e^{ax}$ **[JNTU (K) Feb. 2015 (Set No.4)]**

(*iii*) Show that $\Delta\left(\frac{f_i}{g_i}\right) = \frac{g_i \Delta f_i - f_i \Delta g_i}{g_i \cdot g_{i+1}}$ **[JNTU (K) Oct. 2018 (Set No. 3)]**

(*iv*) Show that $\Delta f_i^2 = (f_i + f_{i+1}) \Delta f_i$.

[JNTU 2006 (Set No.4), (H) May 2016, (K) Oct. 2018 (Set No. 2)]

Solution : Let $y = f(x)$. The first finite forward difference is $\Delta y_k = y_{k+1} - y_k$.

Put $y_k = f(x_k) = f_k$, we get $\Delta f_k = f_{k+1} - f_k$.

The second difference is $\Delta^2 f_k = \Delta(\Delta f_k) = \Delta(f_{k+1} - f_k) = \Delta f_{k+1} - \Delta f_k$ ... (1)

(*i*) $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta^2 f_0 + \Delta^2 f_1 + \Delta^2 f_2 + \Delta^2 f_3 + ... + \Delta^2 f_{n-1}$

$$= (\Delta f_1 - \Delta f_0) + (\Delta f_2 - \Delta f_1) + (\Delta f_3 - \Delta f_2) + (\Delta f_4 - \Delta f_3) + ... + (\Delta f_n - \Delta f_{n-1}),$$ using (1)

$$= \Delta f_n - \Delta f_0$$

(*ii*) Given $f(x) = e^{ax}$, we have $f(x+h) = e^{a(x+h)}$.

Here, h is the step size and $x_{i+1} = x_i + h$

We have to show that $\Delta^n f(x) = (e^{ah} - 1)^n . e^{ax}$.

This can be proved by mathematical induction.

First we shall prove that this is true for $n = 1$.

$$(e^{ah} - 1)^1 e^{ax} = e^{ah} . e^{ax} - e^{ax} = e^{ah+ax} - e^{ax}$$
$$= e^{a(x+h)} - e^{ax} = f(x+h) - f(x) = \Delta f(x)$$

$\therefore\ \Delta f(x_i) = f(x_i + h) - f(x_i)$

Therefore, the result is true for $n = 1$.

Assume that the problem is true for $n - 1$.

Now consider, $\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$

$$= (e^{ah}-1)^{n-1} e^{a(x+h)} - (e^{ah}-1)^{n-1}.e^{ax}$$
$$= (e^{ah}-1)^{n-1}.[e^{a(x+h)} - e^{ax}] = (e^{ah}-1)^{n-1}.[e^{ax+}$$
$$= (e^{ah}-1)^{n-1}.[e^{ax}(e^{ah}-1)] = (e^{ah}-1)^{n-1}.(e^{ah} -$$
$$= (e^{ah}-1)^{n-1+1}.e^{ax} = (e^{ah}-1)^{n}.e^{ax}$$

$\therefore \Delta^n f(x) = (e^{ah}-1)^n . e^{ax}$.

) According to first forward difference, $\Delta\left(\frac{f_i}{g_i}\right) = \frac{f_{i+1}}{g_{i+1}} - \frac{f_i}{g_i}$

Now $\frac{g_i \Delta f_i - g_i \Delta g_i}{g_i \cdot g_{i+1}} = \frac{g_i(f_{i+1} - f_i) - f_i(g_{i+1} - g_i)}{g_i \cdot g_{i+1}}$

$$= \frac{g_i f_{i+1} - g_i f_i - f_i g_{i+1} + g_i f_i}{g_i \cdot g_{i+1}} = \frac{g_i f_{i+1} - f_i g_{i+1}}{g_i \cdot g_{i+1}}$$

$$= \frac{g_i f_{i+1}}{g_i \cdot g_{i+1}} - \frac{f_i g_{i+1}}{g_i \cdot g_{i+1}} = \frac{f_{i+1}}{g_{i+1}} - \frac{f_i}{g_i}$$

$$\therefore\ \Delta\left(\frac{f_i}{g_i}\right) = \frac{g_i \Delta f_i - f_i \Delta g_i}{g_i \cdot g_{i+1}}$$

) We know that $\Delta f_k = f_{k+1} - f_k$

$\therefore\ \Delta f_i^2 = f_{i+1}^2 - f_i^2 = (f_{i+1} + f_i)(f_{i+1} - f_i)$

$= (f_{i+1} + f_i)\Delta f_i \left[\because a^2 - b^2 = (a-b)(a+b)\right]$

Example 11 : If the interval of differencing is unity prove that

$\Delta[x(x+1)(x+2)(x+3)] = 4(x+1)(x+2)(x+3)$ [JNTU 2008 (Se

Solution : Let $f(x) = x(x+1)(x+2)(x+3)$. Then

$\Delta[x(x+1)(x+2)(x+3)] = f(x+h) - f(x)$. Here $h = 1$

Then we have

$$\Delta[x(x+1)(x+2)(x+3)] = (x+1)(x+2)(x+3)(x+4) - x(x+1)(x+2)(x+$$
$$= (x+1)(x+2)(x+3)[x+4-x]$$
$$= 4(x+1)(x+2)(x+3)$$

Example 13 : Show that $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)] = 24 \times 2^{10} \times 10!$ if $h = 2$

[JNTU(H) 2009 (Set No.

Solution : $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$

$= \Delta^{10}[(-1)(-2)(-3)(-4)\, x^{10}$ + terms containing powers of x less than 10]

$= 24\Delta^{10}[x^{10}]$

$= 24 \lfloor 10 \,.\, 2^{10}$ $\quad [\because \Delta^n f(x) = \lfloor n \, h^n$ and $h = 2]$

Example 14 : If $f(x) = x^3 + 5x - 7$, form a table of forward differences taking $x = -1, 0, 1, 2, 3, 4, 5$. Show that the third differences are constant.

Solution : Here $f(-1) = -1 - 5 - 7 = -13$,

$f(0) = 0 - 7 = -7$,

$f(1) = 1 + 5 - 7 = -1$,

$f(2) = 8 + 10 - 7 = 11$,

$f(3) = 27 + 15 - 7 = 35$,

$f(4) = 64 + 20 - 7 = 77$,

$f(5) = 125 + 25 - 7 = 143$

We form the difference table as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
−1	−13			
		6		
0	−7			
		6	0	
1	−1			
		12	6	6
2	11			
		24	12	6
3	35			
		42	18	6
4	77			
		66		
5	143		24	6

We note from the table that all the third forward differences are constan

Example 15 : If y_x is the value of y at x for which the fifth differences and $y_1 + y_7 = -784$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$, find y_4.

Solution : Since fifth differences are constant, $\Delta^6 y_1 = 0$

$\Rightarrow (E - 1)^6 y_1 = 0$

$\Rightarrow (E^6 - 6_{c_1} E^5 + 6_{c_2} E^4 - 6_{c_3} E^3 + 6_{c_4} E^2 - 6_{c_5} E + 6_{c_6} 1)y_1 = 0$

$\Rightarrow y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0$

$\Rightarrow (y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5) - 20y_4 = 0$

$\Rightarrow -784 - 6(686) + 15(1088) - 20y_4 = 0$, using given data

$\Rightarrow -784 - 4116 + 16320 - 20y_4 = 0 \Rightarrow 11420 - 20y_4 = 0$

$\therefore y_4 = 571.$

.8 NEWTON'S FORWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a polynomial of degree n and taken in the following form

$$y = f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ \dots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1}) \quad \dots(A)$$

This polynomial passes through all the points $[x_i, y_i]$ for $i = 0$ to n. Therefore, we can obtain the y_i's by substituting the corresponding x_i's as :

At $x = x_0, y_0 = b_0$

At $x = x_1, y_1 = b_0 + b_1(x_1 - x_0)$

At $x = x_2, y_2 = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$ ...(1)

Let 'h' be the length of interval such that x_i's represent

$x_0, x_0 + h, x_0 + 2h, x_0 + 3h, \dots, x_0 + nh$.

This implies $x_1 - x_0 = h, x_2 - x_0 = 2h, x_3 - x_0 = 3h, \dots, x_n - x_0 = nh$ ...(2)

From (1) and (2), we get

$$y_0 = b_0$$

$$y_1 = b_0 + b_1 h$$

$$y_2 = b_0 + b_1 2h + b_2(2h)h$$

$$y_3 = b_0 + b_1 3h + b_2(3h)(2h)h + b_3(3h)(2h)h$$

....

....

$$y_n = b_0 + b_1(nh) + b_2(nh)(n-1)h + \dots + b_n(nh)[(n-1)h][(n-2)h] \quad \dots(B)$$

Solving the above equations for $b_0, b_1, b_2, \dots, b_n$, we get

$$b_0 = y_0$$

$$b_1 = \frac{y_1 - b_0}{h} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

$$b_2 = \frac{y_2 - b_0 - b_1 2h}{2h^2} = \frac{y_2 - y_0 - \left(\frac{y_1 - y_0}{h}\right)2h}{2h^2}$$

$$= \frac{y_2 - y_0 - 2y_1 - 2y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

$$\therefore\ b_2 = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly, we can see that

$$b_3 = \frac{\Delta^3 y_0}{3!h^3},\ b_4 = \frac{\Delta^4 y_0}{4!h^4}, \ldots,\ b_n = \frac{\Delta^n y_0}{n!h^n}$$

$$\therefore\ y = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)$$

$$(x - x_1)(x - x_2) + \ldots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \quad \ldots(3)$$

If we use the relationship $x = x_0 + ph \Rightarrow x - x_0 = ph$, where $p = 0, 1, 2, \ldots, n$

then $\quad x - x_1 = x - (x_0 + h) = (x - x_0) - h = ph - h = (p - 1)h$

$\quad x - x_2 = x - (x_1 + h) = (x - x_1) - h = (p - 1)h - h = (p - 2)h$

....

....

$\quad x - x_i = (p - i)h$

....

....

$\quad x - x_{n-1} = [p - (n - 1)]h$

$\therefore$ Equation (3) becomes

$$y = f(x) = f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \ldots + \frac{p(p-1)(p-2)\ldots[p-(n-1)]}{n!}\Delta^n y_0 \quad \ldots(4)$$

This formula is known as ***Newton's forward interpolation formula*** (or) ***Newton Gregory forward interpolation formula.***

This is useful for interpolation near the beginning of a set of tabular values.

5.9 NEWTON'S BACKWARD INTERPOLATION FORMULA

If we consider $y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1})$

$+ a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \ldots + a_n(x - x_n)(x - x_{n-1})\ldots(x - x_1)$

and impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_n, x_{n-1},$ $\ldots, x_2, x_1, x_0$, we obtain

NEWTON'S BACKWARD I.F

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \ldots + \frac{p(p+1)\ldots(p+\overline{n-1})}{n!}\nabla^n y_n + \ldots,$$

here $p = \frac{x - x_n}{h}$.

... (6)

This uses tabular values to the left of y_n. Thus this formula is useful for interpolation ear the end of the tabular values.

Formulae for Error in Polynomial Interpolation

If $y = f(x)$ is the exact curve and $y = \phi_n(x)$ is the interpolating polynomial curve, en the error in polynomial interpolation is given by

$$\text{Error} = f(x) - \phi_n(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!} f^{n+1}(\xi) \qquad \ldots (7)$$

r any x, where $x_0 < x < x_n$ and $x_0 < \xi < x_n$.

The error in Newton's forward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p-1)(p-2)\ldots(p-n)}{(n+1)!}\Delta^{n+1} f(\xi) \text{ where } p = \frac{x-x_0}{h} \qquad \ldots(8)$$

The error in Newton's backward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p+1)(p+1)(p+n)}{(n+1)!} h^{n+1} y^{n+1} f(\xi)$$

here $p = \frac{x - x_n}{h}$

...(9)

SOLVED EXAMPLES

Example 1 : The following data gives the melting points of an alloy of lead and zinc.

Percentage of lead in the alloy (p) :	50	60	70	80
Temperature (θ°C) :	205	225	248	274

Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula.

Solution : The difference table is as under :

x	y	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

$P = \frac{x_1 - x_0}{h}$

Let temperature = $f(x)$

We have $x_0 = 50$, $h = 10$ and

$x = x_0 + ph = 54 \Rightarrow 50 + p(10) = 54$ or $p = 0.4$

By Newton's Forward interpolation formula,

$$f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)\,(p-2)}{3!}\Delta^3 y_0 + \dots$$

$$\therefore\ f(54) = 205 + 0.4\,(20) + \frac{0.4\,(0.4-1)}{2!}(3) + \frac{0.4\,(0.4-1)\,(0.4-2)}{3!}(0)$$

$$= 205 + 8 - 0.36 = 212.64\,.$$

Hence melting point = 212.64

Example 2 : State appropriate interpolation formula which is to be used to calc the value of exp(1.75) from the following data and hence evaluate it from the given d

x	1.7	1.8	1.9	2.0
$y = e^x$	5.474	6.050	6.686	7.389

[JNTU (A) June 2013 (Set No

Solution : The difference table is as under :

x	y	Δ	Δ^2	Δ^3
1.7	5.474			
		0.576		
1.8	6.050		0.060	
		0.636		0.007
1.9	6.686		0.067	
		0.703		
2.0	7.389			

Let $f(x) = y = e^x$. Then we have $x_0 = 1.7,\ h = 0.1$ and $x_0 + ph = 1.75$

$\Rightarrow 1.7 + p(0.1) = 1.75$ or $p = 0.5$

By Newton's Forward interpolation formula,

$$f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)\,(p-2)}{\underline{|3}}\Delta^3 y_0 + \dots$$

$$\therefore\ f(1.75) = 5.474 + 0.5 \times (0.576) + \frac{0.5\,(0.5-1)}{2}(0.060)$$

$$+ \frac{0.5\,(0.5-1)\,(0.5-2)}{6}(0.007)$$

$$= 5.474 + 0.288 - 0.0075 + 0.0004375 = 5.7624375 - 0.0075 = 5.754937$$

$$= 5.7549 \text{ (Rounded up to four decimal places).}$$

Example 3 : Applying Newton's forward interpolation formula, compute the val $\sqrt{5.5}$, given that $\sqrt{5} = 2.236, \sqrt{6} = 2.449, \sqrt{7} = 2.646$ and $\sqrt{8} = 2.828$ correct three places of decimal.

Solution : Let $f(x) = \sqrt{x}$.

The difference table is

x	y	Δ	Δ^2	Δ^3
5	2.236			
		0.213		
6	2.449		-0.016	
		0.197		0.001
7	2.646		-0.015	
		0.182		
8	2.828			

We have

$x_0 + ph = 5.5,\ x_0 = 5,\ h = 1$

$\Rightarrow\ 5 + p(1) = 5.5 \quad \text{or} \quad p = 0.5$

By Newton's Forward interpolation formula,

$$f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\underline{|3}}\Delta^3 y_0 + \ldots$$

$$\therefore\ f(5.5) = 2.236 + 0.5 \times (0.213) + \frac{0.5(0.5-1)}{\underline{|2}}(-0.016) + \frac{0.5(0.5-1)(0.5-2)}{\underline{|3}}(0.001)$$

i.e. $\sqrt{5.5} = 2.236 + 0.1065 + 0.00200 + 0.0000625$

$= 2.3445625 = 2.345$ (Rounded upto four decimal places).

Example 4 : If $\mu_0 = 1, \mu_1 = 0, \mu_2 = 5, \mu_3 = 22, \mu_4 = 57$ find $\mu_{0.5}$.

Solution : The difference table is

x	μ_x	Δ	Δ^2	Δ^3	Δ^4
0	1				
		-1			
1	0		6		
		5		6	
2	5		12		0
		17		6	
3	22		18		
		35			
4	57				

We have $x_0 + ph = 0.5,\ x_0 = 0,\ h = 1$

$\Rightarrow\ 0 + p(1) = 0.5 \quad \text{or} \quad p = 0.5$

By Newton's Forward interpolation formula,

$$\mu_{0.5} = \mu_0 + 0.5\,\Delta\mu_0 + \frac{0.5\,(0.5-1)}{\lfloor 2}\Delta^2\mu_0 + \frac{0.5\,(0.5-1)\,(0.5-2)}{\lfloor 3}\Delta^3\mu_0$$

$$= 1 + (0.5)\,(-1) + \frac{0.5\,(-0.5)}{2}6 + \frac{0.5\,(-0.5)\,(-1.5)}{6}6$$

$$= 1 - 0.5 - 0.75 + 0.375 = 0.125.$$

Example 5 : Using Newton's forward interpolation formula, and the given table values

x	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

Obtain the value of $f(x)$ when $x = 1.4$. [JNTU (A) June 2013 (Set No...)]

Solution : The difference table is

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
1.1	0.21				
		0.48			
1.3	0.69		0.08		
		0.56		0	
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

If we take $x_0 = 1.3$, then $y_0 = 0.69$, $\Delta y_0 = 0.56$, $\Delta^2 y_0 = 0.08$, $\Delta^3 y_0 = 0$

$h = 0.2, x = 1.3$

We have $x_0 + ph = 1.4 \Rightarrow 1.3 + p\,(0.2) = 1.4 \Rightarrow p = \frac{1}{2}$

Using Newton's interpolation formula,

$$f(1.4) = 0.69 + \frac{1}{2} \times 0.56 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{\lfloor 2} \times 0.08$$

$$= 0.69 + 0.28 - 0.01 = 0.96$$

Note : $x_0 = 1.3$ is taken so that $h < 1$.

Example 6 : Find the Newton's forward difference interpolating polynomial for the data :

x	0	1	2	3
$f(x)$	1	3	7	13

Solution : The difference table is

x	$f(x)$	Δ	Δ^2	Δ^3
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

By Newton's Forward interpolation formula,

$$f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)\,(p-2)}{\lfloor 3}\Delta^3 y_0 + \ldots$$

Here $x_0 = 0$, $n = 1$ and $p = x$

Thus we have $f(x) = 1 + x\,(2) + \dfrac{x\,(x-1)}{\lfloor 2}(2) + \dfrac{x\,(x-1)\,(x-2)}{\lfloor 3}(0) + \ldots$

$$= 1 + 2x + x^2 - x = x^2 + x + 1.$$

Example 7 : The following table gives corresponding values of x and y. Construct the difference table and then express y as a function of x :

x	0	1	2	3	4
y	3	6	11	18	27

Solution : The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	3				
		3			
1	6		2		
		5		0	
2	11		2		0
		7		0	
3	18		2		
		9			
4	27				

We have

$$x_0 + ph = x,\ x_0 = 0,\ h = 1$$

$\Rightarrow 0 + p(1) = x$ or $p = x$

By Newton's forward interpolation formula,

$$f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)\,(p-2)}{\lfloor 3}\Delta^3 y_0 + \ldots$$

i.e. $f(x) = 3 + x\,(3) + \dfrac{x\,(x-1)}{\lfloor 2}(2) + \dfrac{x\,(x-1)\,(x-2)}{\lfloor 3}(0) + \ldots$

i.e. $f(x) = 3 + 3x + x^2 - x + 0$ or $f(x) = x^2 + 2x + 3$.

Example 8 : Consider the following data for $g(x) = \frac{\sin x}{x^2}$

x	0.1	0.2	0.3	0.4	0.5
$g(x)$	9.9833	4.9696	3.2836	2.4339	1.9177

[JNTUA 20...

Calculate $g(0.25)$ accurately using Newton's forward method of interpolation.

Solution : Newton's Forward interpolation formula is

$$f(x) = f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \ldots$$

Let $x = x_0 + ph$. Here $x = 0.25,\ x_0 = 0.1$

Step interval $h = 0.2 - 0.1 = 0.1$

$$\therefore\ p = \frac{x - x_0}{h} = \frac{0.25 - 0.1}{0.1} = \frac{0.15}{0.1} = 1.5$$

The Newton's forward difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	9.9833				
		−5.0137			
0.2	4.9696		3.3277		
		−1.6860		−2.4914	
0.3	3.2836		0.8363		1.9886
		−0.8497		−0.5028	
0.4	2.4339		0.3335		
		−0.5162			
0.5	1.9177				

$$g(0.25) = 9.9833 + 1.5(-5.0137) + \frac{1.5 \times 0.5}{2} \times 3.3277 + \frac{1.5 \times 0.5 \times (-0.5)}{3 \times 2} \times (-2.4919)$$

$$+ \frac{1.5 \times 0.5 \times (-0.5) \times (-1.5)}{4 \times 3 \times 2} \times 1.9886$$

$$= 9.9833 - 7.52 + 1.24789 + 0.1557 + 0.0466 = 3.9135$$

$\therefore\ g(0.25) = 3.9135$

Example 9 : For $x = 0, 1, 2, 3, 4$; $f(x) = 1, 14, 15, 5, 6$. Find $f(3)$ using For... difference table.

[JNTU 2004, (A) June 2011, (K) Dec. 2016 (Set N...

Solution : Given

x	0	1	2	3	4
$f(x)$	1	14	15	5	6

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		13			
1	14				
			−12		
		1			
2	15			1	
			−11		21
		−10			
3	5			22	
			−11		
		1			
4	6				

Let $x_0 = 0$, $x = 3, h = 1$. Then $p = \dfrac{x - x_0}{h} = \dfrac{3-0}{1} = 3$

From the above table, we have

$\Delta y_0 = 13$, $\Delta^2 y_0 = -12$, $\Delta^3 y_0 = 1, \Delta^4 y_0 = 21$, $\Delta y_1 = 1$,

$\Delta^2 y_1 = -11$, $\Delta^3 y_1 = 22$, $\Delta y_2 = -10$, $\Delta^2 y_2 = 11$

By Newton's forward interpolation formula,

$$f(x) = f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)\,(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + 13\,(3) + \frac{3\,(2)}{2}(-12) + \frac{3\,(2)\,(1)}{3 \times 2 \times 1}(1)$$

$$= 5.$$

Example 10 : Find the cubic polynomial which takes the following values :

$y(0) = 1, y(1) = 0, y(2) = 1$ and $y(3) = 10$. Hence, or otherwise, obtain $y(4)$.

Solution : We form the difference table as :

x	y	Δ	Δ^2	Δ^3
0	1			
		−1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

Here $h = 1$. Hence, take $x = x_0 + ph$ and $x_0 = 0$. We obtain $p = x$.

Substituting the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ and p in Newton's forward interpolation formula, we get

$$y(x) = 1 + x(-1) + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{(6)}(6) = x^3 - 2x^2 + 1$$

which is the polynomial form which we obtained the above tabular values. To compute we observe that $p = 4$. Hence formula gives $y(4) = 1 + 4(-1) + (12) + 24 = 33$ which the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

Note. This process of finding the value of y for some value of x outside the given called **extrapolation** and this example demonstrates the fact that if a tabulated function polynomial, then interpolation and extrapolation would give exact values.

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Example 15 : Find $f(2.5)$ using Newtons forward formula from the following table

x	0	1	2	3	4	5	6
y	0	1	16	81	256	625	1296

[JNTU May 2006 (Set No

Solution : We have $x = 2.5, h = 1, p = \frac{x - x_0}{h} = \frac{2.5 - 0}{1} = 2.5$

$\Delta y_0 = y_1 - y_0 = 1 - 0 = 1$

$\Delta y_1 = y_2 - y_1 = 16 - 1 = 15$

$\Delta y_2 = y_3 - y_2 = 81 - 16 = 65$

$\Delta y_3 = y_4 - y_3 = 256 - 81 = 175$

$\Delta y_4 = y_5 - y_4 = 1296 - 625 = 671$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = 15 - 1 = 14$

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = 65 - 15 = 50$

$\Delta^2 y_2 = \Delta y_3 - \Delta y_2 = 175 - 65 = 110$

$\Delta^2 y_3 = \Delta y_4 - \Delta y_3 = 674 - 175 = 499$

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = 50 - 14 = 36$

$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1 = 110 - 50 = 60$

$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2 = 499 - 110 = 389$

$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = 60 - 36 = 24$

$\Delta^4 y_1 = \Delta^3 y_2 - \Delta^3 y_1 = 389 - 60 = 329$

$\Delta^5 y_0 = \Delta^4 y_1 - \Delta^4 y_0 = 329 - 24 = 305$

$p = \frac{x - x_0}{h}$

Using Newton Forward Difference Formula, we have

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{\lfloor 4}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{\lfloor 5}\Delta^5 y_0$$

$$\therefore \quad f(2.5) = 0 + 2.5(1) + \frac{(2.5)(1.5)}{\lfloor 2}(14) + \frac{(2.5)(1.5)(.5)}{\lfloor 3}(36) + \frac{(2.5)(1.5)(.5)(-.5)}{\lfloor 4}(24)$$

$$+ \frac{(2.5)(1.5)(.5)(-.5)(-1.5)}{\lfloor 5}(305)$$

$$= 2.5 + 26.25 + 11.25 - 0.9375 + 3.5390 = 42.6015.$$

Example 16 : Find $y(1.6)$ using Newton's Forward difference formula from the table

x	1	1.4	1.8	2.2
y	3.49	4.82	5.96	6.5

[JNTU May 2006, (K) Dec. 2016 (Set No. 4)]

Solution : Let $x_0 = 1, h = 1.4 - 1 = .4, x_0 + ph = 1.6 \Rightarrow 1 + .4p = 1.6 \Rightarrow p = \frac{.6}{.4} = \frac{3}{2}$

We have $\Delta y_0 = y_1 - y_0 = 4.82 - 3.49 = 1.33$

$\Delta y_1 = y_2 - y_1 = 5.96 - 4.82 = 1.14$

$\Delta y_2 = y_3 - y_2 = 6.5 - 5.96 = .54$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = 1.14 - 1.33 = -0.19$

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = .54 - 1.14 = -.60$

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = -0.60 + 0.19 = -0.41.$

Using Newton's forward difference formula, we have

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0$$

i.e. $$f(1.6) = 3.49 + \frac{3}{2}(1.33) + \frac{\left(\frac{3}{2}\cdot\frac{1}{2}\right)(-0.19)}{\lfloor 2} + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(-0.41)}{\lfloor 3}$$

$$= 3.49 + 1.995 - 0.07125 + 0.025625$$

$$= 5.4394.$$

Example 17 : Construct difference table for the following data.

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate $f(0.6)$. [JNTU May 2007 (Set N

Solution : The difference table is

x	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
0.1	0.003			
		0.064		
0.3	0.067		0.017	
		0.081		0.002
0.5	0.148		0.019	
		0.1		0.003
0.7	0.248		0.022	
		0.122		0.004
0.9	0.370		0.026	
		0.148		0.005
1.1	0.518		0.031	
		0.179		
1.3	0.697			

Here $x = 0.6, x_0 = 0.1, h = 0.2, y_0 = 0.003, \Delta y_0 = 0.064, \Delta^2 y_0 = 0.017, \Delta^3 y_0 = 0.002$

We have $x_0 + ph = x$

$\Rightarrow \quad 0.1 + p(0.2) = 0.6 \Rightarrow p(0.2) = 0.5$

$\Rightarrow \quad p = \dfrac{0.5}{0.2} \quad \therefore p = 2.5$

By Newton's forward difference formula,

$$y(x) = f(x_0 + ph) = y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_0) + ...$$

$$i.e., f(0.6) = 0.003 + (2.5)(0.064) + \frac{(2.5)(2.5-1)}{2}(0.017) + \frac{(25)(2.5-1)(2.5)(0.002)}{6}$$

$$= 0.003 + 0.16 + 0.031875 + 0.000625 = 0.1955$$

$\therefore f(0.6) = 0.1955.$

Example 18 : (*i*) Find $y(55)$ given that $y(50) = 205, y(60) = 225, y(70) = 248$ and $y(30) = 274$, using Newton's forward difference formula.

[JNTU (H) Jan. 2012, (K) Dec. 2015 (Set No. 2)]

(*ii*) Find $y(66)$ given that $y(50) = 201, y(60) = 225, y(70) = 248$ and $y(80) = 274$ using Newton's backward difference formula. **[JNTU (K) Dec. 2015 (Set No. 1)]**

Solution : (*i*) We are given

x	50	60	70	80
$y(x)$	205	225	248	274

Here, $h = 10, x_0 = 50, x_0 + ph = 55 \Rightarrow p = \dfrac{55-50}{10} = 0.5$

The difference table is

x	$y(x)$	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Using Newton's forward difference formula,

$$y(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p_{-1})}{2!}\Delta^2 y_0 + \frac{p(p_{-1})(p_{-2})}{3!}\Delta^3 y_0$$

$$\therefore y(55) = 205 + (0.5)(20) + \frac{(0.5)(-0.5)}{2}(3)$$

$$= 205 + 10 - 0.375 = 215 - 0.375 = 214.625$$

(*ii*) This is left as an exercise to the reader.

5.10 CENTRAL DIFFERENCE INTERPOLATION

As mentioned earlier, Newton's forward interpolation formula is useful to find the of $y = f(x)$ at a point which is near the beginning value of x and the Newton's back interpolation formula is useful to find the value of 'y' at a point which is near the ter value of x. We now derive the interpolation formulas that can be employed to find the of x which is around the middle to the specified values.

For this purpose, we take x_0 as one of the specified values of x that lies around the middle of the difference table and denote $x_0 - rh$ by x_{-r} and the corresponding value of y by y_{-r}. Then we can write the difference table in the two notations as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2}	y_{-2}				
		Δy_{-2} $(= \delta y_{-3/2})$			
x_{-1}	y_{-1}				
			$\Delta^2 y_{-2}$ $(= \delta^2 y_{-1})$		
		Δy_{-1} $(= \delta y_{-1/2})$		$\Delta^3 y_{-2}$ $(= \delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1}$ $(= \delta^2 y_0)$		$\Delta^4 y_{-2}$ $(= \delta^4 y_0)$
		Δy_0 $(= \delta y_{1/2})$		$\Delta^3 y_{-1}$ $(= \delta^3 y_{1/2})$	
x_1	y_1		$\Delta^2 y_0$ $(= \delta^2 y_1)$		
		Δy_1 $(= \delta y_{3/2})$			
x_2	y_2				

From the table, we note the following :

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1},\ \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1},\ \Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1},$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \qquad \text{...(1)}$$

and so on.

Also, $\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2}$, $\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2}$, $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$,

$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$, $\Delta^5 y_{-1} = \Delta^5 y_{-2} + \Delta^6 y_{-2}$ and so on. ...(2)

By using the expressions (1) and (2), we now obtain two versions of the following Newton's Forward interpolation formula :

SOLVED EXAMPLES

Example 1 : Evaluate $f(10)$ given $f(x) = 168, 192, 336$ at $x = 1, 7, 15$ respectively. Use Lagrange interpolation. [JNTU 2002, (A) May 2012 (Set No. 2)]

Solution : We are given

$x_0 = 1, x_1 = 7, x_2 = 15, x = 10$ and

$y_0 = 168, y_1 = 192, y_2 = 336, y = ?$

The Lagrange's formula is

$$y = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

On substitution, we have

$$y = f(10) = \frac{(10-7)(10-15)}{(1-7)(1-15)} \times 168 + \frac{(10-1)(10-15)}{(7-1)(7-15)} \times 192 + \frac{(10-1)(10-7)}{(15-1)(15-7)} \times 336$$

$$= \frac{-15}{84} \times 168 + \frac{-45}{-48} \times 192 + \frac{27}{112} \times 336$$

$$= -0.1786 \times 168 + 0.9375 \times 192 + 0.24 \times 336$$

$$= -30.005 + 180 + 81.01 = 231.005 \text{ approx.}$$

Example 2 : Using Lagrange formula, calculate $f(3)$ from the following table.

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

[JNTU May 03]

Solution : Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 5, x_5 = 6$

and $f(x_0) = 1, f(x_1) = 14, f(x_2) = 15, f(x_3) = 5, f(x_4) = 6, f(x_5) = 19$

From Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} f(x_2) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} f(x_3) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} f(x_4) + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(x_5)$$

Here $x = 3$.

$$\therefore\ f(3) = \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \times 1 + \frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14 +$$

$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15 + \frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5 +$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 6 + \frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 19$$

$$= \frac{12}{240} - \frac{18}{60} \times 14 + \frac{36}{48} \times 15 + \frac{36}{48} \times 5 - \frac{18}{60} \times 6 + \frac{12}{240} \times 19$$

$$= 0.05 - 4.2 + 11.25 + 3.75 - 1.8 + 0.95 = 10$$

Hence $f(3) = 10$.

Example 3 : Using Lagrange's interpolation formula, find the value of $y(10)$ from the following table:

x	5	6	9	11
y	12	13	14	16

[JNTU Aug. 2008S, (K) Feb. 2015, May 2016 (Set No.

(or) Find $y(10)$, Given that $y(5) = 12$, $y(6) = 13$, $y(9) = 14$, $y(11) = 16$ using Lagrange formula.

[JNTU(H) June 2010 (Set No.

Solution : Lagrange's interpolation formula is given by

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2)$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3) + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

Given $x_1 = 5, x_2 = 6, x_3 = 9, x_4 = 11$

Here $x = 10$, $f(x_1) = 12$, $f(x_2) = 13$, $f(x_3) = 14$, $f(x_4) = 16$

$$\therefore\ f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}\times 14+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times 16$$

$$=\frac{4\times 1\times -1}{-1\times -4\times -6}\times 12+\frac{5\times 1\times -1}{1\times -3\times -5}\times 13+\frac{5\times 4\times -1}{4\times 3\times -2}\times 14+\frac{5\times 4\times 1}{6\times 5\times 2}\times 16$$

$$=2-\frac{13}{3}+\frac{35}{3}+\frac{16}{3}=14\frac{2}{3}=14.6666.$$

Aliter : Refer Solved Example 15.

Example 4 : Given $u_0 = 580$, $u_1 = 556$, $u_2 = 520$ and $u_4 = 385$ find u_3.

Solution : Given data can be tabulated as follows.

x	0	1	2	4
$u(x)$	580	556	520	385

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_4 = 3$ and

$f(x_0) = f(0) = u_0 = 580$; $f(x_1) = f(1) = u_1 = 556$

$f(x_2) = f(2) = u_2 = 520$; $f(x_4) = f(4) = u_4 = 385$

By Lagrange's formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_4)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_4)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_4)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)}f(x_4)$$

$$\therefore f(3)=\frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)}(580)+\frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)}(556)$$
$$+\frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)}(520)+\frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)}(385)$$

$$=\frac{2\times 1\times -1}{-1\times -2\times -4}(580)+\frac{3\times 1\times -1}{1\times -1\times -3}(556)+\frac{3\times 2\times -1}{2\times 1\times -2}(520)+\frac{3\times 2\times 1}{4\times 3\times 2}(385)$$

$$=145-556+780+96.25=465.25.$$

Example 5 : The values of a function $f(x)$ are given below for certain values of x

x	0	1	3	4
$f(x)$	5	6	50	105

Find the values of $f(2)$ using Lagrange's interpolation formula.

Solution : By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1)+\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2)$$
$$+\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}f(x_3)+\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)$$

$$\therefore f(2) = \frac{(2-1)(2-3)(2-4)}{(0-1)(0-3)(0-4)}(5) + \frac{(2-0)(2-3)(2-4)}{(1-0)(1-3)(1-4)}(6)$$
$$+ \frac{(2-0)(2-1)(2-4)}{(3-0)(3-1)(3-4)}(50) + \frac{(2-0)(2-1)(2-3)}{(4-0)(4-1)(4-3)}(105)$$
$$= \frac{1\times -1\times -2}{-1\times -3\times -4}(5) + \frac{2\times -1\times -2}{1\times -2\times -3}(6) + \frac{2\times 1\times -2}{3\times 2\times -1}(50) + \frac{2\times 1\times -1}{4\times 3\times 1}(105)$$
$$= \frac{-5}{6} + 4 + \frac{100}{3} - \frac{35}{2} = \frac{-5+24+200-105}{6} = \frac{114}{6} = 19.$$

Example 6 : Given the values :

x	0	2	3	6
$f(x)$	–4	2	14	158

Using Lagrange's formula for interpolation find the value of $f(4)$.

Solution : Using Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2)$$
$$+ \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}f(x_3) + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)$$

Here $x = 4$, $x_1 = 0, x_2 = 2, x_3 = 3, x_4 = 6$

and $f(x_1) = -4, f(x_2) = 2, f(x_3) = 14, f(x_4) = 158$

$$\therefore f(4) = \frac{(4-2)(4-3)(4-6)}{(0-2)(0-3)(0-6)}(-4) + \frac{(4-0)(4-3)(4-6)}{(2-0)(2-3)(2-6)}(2)$$
$$+ \frac{(4-0)(4-2)(4-6)}{(3-0)(3-2)(3-6)}(14) + \frac{(4-0)(4-2)(4-3)}{(6-0)(6-2)(6-3)}(158)$$
$$= \frac{2\times 1\times(-2)}{-2\times -3\times -6}(-4) + \frac{4\times 1\times(-2)}{2\times -1\times -4}(2) + \frac{4\times 2\times -2}{3\times 1\times -3}(14) + \frac{4\times 2\times 1}{6\times 4\times 3}(158)$$
$$= \frac{-4}{9} - 2 + \frac{224}{9} + \frac{158}{9} = \frac{-4-18+224+158}{9} = 40.$$

Example 7 : State Lagrange's formula of interpolation, using unequal intervals. From an experiment, we get the following values of a function $f(x)$:

x	1	2	–4
$f(x)$	3	–5	4

Represent the function $f(x)$ approximately by a polynomial of degree 2.

Solution : By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}f(x_1) + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}f(x_2)$$
$$+ \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}f(x_3)$$

Here $x_1 = 1, x_2 = 2, x_3 = -4$; $f(x_1) = 3, f(x_2) = -5, f(x_3) = 4$

$$\therefore\ f(x) = 3 \times \frac{(x-2)(x+4)}{(1-2)(1+4)} + (-5)\frac{(x-1)(x+4)}{(2-1)(2+4)} + 4 \times \frac{(x-1)(x-2)}{(-4-1)(-4-2)}$$

$$= \frac{-3}{5}(x^2 + 2x - 8) - \frac{5}{6}(x^2 + 3x - 4) + \frac{4}{30}(x^2 - 3x + 2)$$

$$= \left(\frac{-3}{5} - \frac{5}{6} + \frac{4}{30}\right)x^2 + \left(\frac{-6}{5} - \frac{15}{6} - \frac{4}{10}\right)x + \left(\frac{24}{5} + \frac{10}{3} + \frac{4}{15}\right)$$

$$\therefore\ f(x) = \frac{-13}{10}x^2 - \frac{41}{10}x + \frac{42}{5} = \frac{-1}{10}(13x^2 + 41x - 84).$$

Example 8 : Find the interpolation polynomial for the following :

x	0	1	2	5
$f(x)$	2	3	12	147

Solution : By Lagrange's interpolation formula,

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= \frac{-1}{5}(x^3 - 8x^2 + 17x - 10) + \frac{3}{4}(x^3 - 7x^2 + 10x) - 2(x^3 - 6x^2 + 5x)$$

$$+ \frac{49}{20}(x^3 - 3x^2 + 2x)$$

$$= \frac{1}{20}(-4x^3 + 15x^3 - 40x^3 + 49x^3) + \frac{1}{20}(32x^2 - 105x^2 + 240x^2 - 147x^2)$$

$$+ \frac{1}{20}(-68x + 150x - 200x + 98x) + 2$$

$$= x^3 + x^2 - x + 2$$

Example 22 : Using Lagrange's formula find y (6) given:

x	3	5	7	9	11
y	6	24	58	108	74

[illegible]

Solution Given $x_0 = 3, x_1 = 5, x_2 = 7, x_3 = 9, x_4 = 11$ and

$y_0 = 6, y_1 = 24, y_2 = 58, y_3 = 108, y_4 = 74$

$$\text{Now } f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4$$

Here, $x = 6$

$$\therefore f(6) = \frac{(6-5)(6-7)(6-9)(6-11)}{(3-5)(3-7)(3-9)(3-11)}(6) + \frac{(6-3)(6-7)(6-9)(6-4)}{(5-3)(5-7)(5-9)(5-11)}(24)$$

$$+\frac{(6-3)(6-5)(6-9)(6-11)}{(7-3)(7-5)(7-9)(7-11)}(58) + \frac{(6-3)(6-5)(6-7)(6-11)}{(9-3)(9-5)(9-7)(9-11)}(108)$$

$$+\frac{(6-3)(6-5)(6-7)(6-9)}{(9-3)(9-5)(9-7)(9-11)}(74)$$

$$= \frac{(1)(-1)(-3)(-5)}{(-2)(-4)(-6)(-8)}(6) + \frac{(3)(-1)(-3)(-5)}{(2)(-2)(-4)(-6)}(24) + \frac{(3)(1)(-3)(-5)}{(4)(2)(-2)(-4)}(58)$$

$$+\frac{(3)(1)(-1)(-5)}{(6)(4)(2)(-2)}(108) + \frac{(3)(1)(-1)(-3)}{(6)(4)(2)(-2)}(74)$$

$$= \frac{-15}{-64} + \frac{-45}{-4} + \frac{45}{64}\times(58) + \frac{15}{-96}\times(108) + \frac{9}{96}(74)$$

$= .2343 + 11.25 + 40.7812 - 16.875 + 6.9375$

$= 43.328$

Example 23 : Find $y(5)$ given that $y(0)=1, y(1)=3, y(3)=13$, and $y(8)=123$ using Lagrange's formula. [illegible]

Solution : Given

x	0	1	3	8
y	1	3	13	128

Using Lagrange's formula,

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Take $x_0=0, x_1=1, x_2=3, x_3=8$ and

$y_0=1, y_1=3, y_2=13, y_3=128$

$$\therefore y(5)=\frac{(5-1)(5-3)(5-8)}{(0-1)(0-3)(0-8)}(1)+\frac{(5-0)(5-3)(5-8)}{(1-0)(1-3)(1-8)}(3)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$=\frac{(4)(2)(-3)}{(-1)(-3)(-8)}(1)+\frac{(5)(-2)(-3)}{(1)(-2)(-7)}(3)+\frac{(5)(4)(-3)}{(3)(2)(-5)}(13)+\frac{(5)(4)(2)}{(8)(7)(5)}(128)$$

$$=1+\frac{45}{7}+26+\frac{128}{7}=1+6.4285+26+18.2857=51.7142$$

$\therefore y(5)=51.7142$

Example 24 : Given that $y(3)=6, y(5)=24, y(7)=58, y(9)=108, y(11)=174$ find x when $y=100$, Using Lagranges formula. [JNTU (H) Jan. 2012 (Set No. 2)]

Solution : Here we will view x as a function of y.

y	6	24	58	108	174
x	3	5	7	9	11

By Lagrange's formula,

$$x=f(y)=\frac{(y-y_2)(y-y_3)(y-y_4)(y-y_5)}{(y_1-y_2)(y_1-y_3)(y_1-y_4)(y_1-y_5)}f(y_1)$$

$$+\frac{(y-y_1)(y-y_3)(y-y_4)(y-y_5)}{(y_2-y_1)(y_2-y_3)(y_2-y_4)(y_2-y_5)}f(y_2)$$

$$+\frac{(y-y_1)(y-y_2)(y-y_4)(y-y_5)}{(y_3-y_1)(y_3-y_2)(y_3-y_4)(y_3-y_5)}f(y_3)$$

$$+\frac{(y-y_1)(y-y_2)(y-y_3)(y-y_5)}{(y_4-y_1)(y_4-y_2)(y_4-y_3)(y_4-y_5)}f(y_4)$$

$$+\frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_5-y_1)(y_5-y_2)(y_5-y_3)(y_5-y_4)}$$

Taking $y=100$ and substituting the values, we get

$$x=\frac{(100-24)(100-58)(100-108)(100-174)}{(6-24)(6-58)(6-108)(6-174)}(3)$$

$$+\frac{(100-6)(100-58)(100-108)(100-174)}{(24-6)(24-58)(24-108)(24-174)}(5)$$

$$+\frac{(100-6)(100-24)(100-108)(100-174)}{(58-6)(58-24)(58-108)(58-174)}(7)$$

$$+\frac{(100-6)(100-24)(100-58)(100-174)}{(108-6)(108-24)(108-58)(108-174)}(9)$$

$$+\frac{(100-6)(100-24)(100-58)(100-108)}{(174-6)(174-24)(174-58)(174-108)}(11)$$

$$=\frac{(76)(42)(-8)(-74)}{(-18)(-52)(-102)(-168)}(3)+\frac{(94)(42)(-8)(-74)}{(18)(-34)(-84)(-150)}(5)+\frac{(94)(76)(-8)(-74)}{(52)(34)(-50)(-116)}(7)$$

$$+\frac{(94)(76)(42)(-74)}{(102)(84)(50)(-66)}(9)+\frac{(94)(76)(42)(-8)}{(168)(150)(116)(66)}(11)$$

$$=\frac{1889664}{16039296}\times 3-\frac{2337216}{7711200}\times 5+\frac{4229248}{10254400}\times 7+\frac{22203552}{28204400}\times 9+\frac{2400384}{192931200}\times 11$$

$$=0.3534-1.5154+2.8870+7.0675-0.1368$$

$$=10.3079-1.6522$$

$$=8.6557$$

5.11 NEWTON'S DIVIDED DIFFERENCE INTERPOLATION FORMULA FOR UNEQUAL INTERVALS.

Let $y_0, y_1, y_2, \dots, y_n$ be the values of y corresponding to $x_0, x_1, x_2, \dots, x_n$ of x. (let $y = f(x)$)
By definition of divided differences, we have

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f(x) = f(x_0) + (x - x_0) f(x, x_0) \qquad \dots (1)$$

Similarly $f(x, x_0, x_1) = \dfrac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$

$$\Rightarrow f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)$$

Substituting the values of $f(x, x_1)$ in (1) we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x, x_0, x_1) \qquad \dots (2)$$

Again $f(x, x_0, x_1, x_2) = \dfrac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2}$

$$\Rightarrow f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2)$$

Substituting the value of $f(x, x_0, x_1)$ in (2), we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x, x_0, x_1, x_2) \qquad \dots (3)$$

Proceeding in this way, we get

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f(x_0, x_1) \\ &+ (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \\ &+ \dots\dots\dots\dots\dots \\ &+ (x - x_0)(x - x_1)(x - x_2) \dots (x - x_{n-1}) f(x_0, x_1, x_2, \dots, x_n) \\ &+ (x - x_0)(x - x_1) \dots (x - x_n) f(x, x_0, x_1, \dots, x_n) \qquad \dots (4) \end{aligned}$$

By the property of divided differences, if $f(x)$ is a polynomial of degree n, then $(n+1)^{th}$ vided differences is zero *i.e.*, $f(x, x_0, x_1, \dots, x_n) = 0$.

Therefore, equation (4) becomes

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \dots\dots\dots \\ &+ (x - x_0)(x - x_1) \dots (x - x_{n-1}) f(x_0, x_1, \dots, x_n) \qquad \dots (5) \end{aligned}$$

Equation (5) is called **Newton's divided difference formula for unequal intervals (OR) Newton's general interpolation formula with divided differences.**

Equation (5) can be reduced to Newton's Forward Interpolation formula for equal intervals.

We know $f(x_0, x_1) = \dfrac{\Delta f(x_0)}{1!h}$

$$f(x_0, x_1, x_2) = \frac{\Delta^2 f(x_0)}{2!h^2}$$

$$f(x_0, x_1, x_2, x_3) = \frac{\Delta^3 f(x_0)}{3!h^3}$$

....

$$f(x_0, x_1, x_3, \ldots x_n) = \frac{\Delta^n f(x_0)}{n!h^n}$$

If $x - x_0 = ph$ then $x - x_1 = (x - x_0) - (x_1 - x_0) = ph - h = (p - 1)h$

Similarly, $x - x_2 = (p - 2)h$ etc...

Substituting the above values in equation (5), we get

$$f(x) = f(x_0 + ph) = f(x_0) + ph\frac{\Delta f(x_0)}{h} + \frac{ph(p-1)h}{2!h^2}\Delta^2 f(x_0) + \cdots$$

$$= f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \cdots \qquad \ldots (6)$$

Equation (6) is called Newton's forward interpolation formula for equal intervals.

SOLVED EXAMPLES

Example 1 : Find third divided differences of $f(x)$ with arguments 2, 4, 9, 10 where $f(x) = x^3 - 2x$

Solution :

x	$f(x)$	I order divided difference	II order divided difference	III order divided difference
2	4			
		$\frac{56-4}{4-2} = 26$		
4	56		$\frac{131-26}{9-2} = 15$	
		$\frac{711-56}{9-4} = 131$		$\frac{23-15}{10-2} = 1$
9	711		$\frac{269-131}{10-4} = 23$	
		$\frac{980-711}{10-9} = 269$		
10	980			

x	$f(x)$	I order divided difference	II order divided difference	III order divided difference	IV order divided difference
4	48				
		$\frac{100-48}{5-4}=52$			
5	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
7	294		$\frac{202-97}{10-5}=21$		
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	0
10	900		$\frac{310-202}{11-7}=27$		
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	0
11	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

By Newton's divided difference interpolation formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \ldots \quad \ldots (1)$$

Here $x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_3 = 10$, $x_4 = 11$, $x_5 = 13$ and $f(x_0) = 48$; $f(x_0, x_1) = 5$ $f(x_0, x_1, x_2) = 15$, $f(x_0, x_1, x_2, x_3) = 1$

Hence using these values in (1), we have

$$f(x) = 48 + (x-4)\,52 + (x-4)(x-5)\,15 + (x-4)(x-5)(x-7)\,(1)$$

$$\therefore\ f(2) = 48 - 104 + 90 - 30 = 4$$

$$f(8) = 48 + (4)(52) + (4)(3)\,15 + (4)(3)(1) = 448$$

$$f(15) = 48 + 11 \times 52 + 11 \times 10 \times 15 + 11 \times 10 \times 8 = 3150$$

Example 4 : If $f(x) = \frac{1}{x^2}$ find the divided differences $f(a, b)$; $f(a, b, c)$ and $f(a, b, c,$

Solution : $f(a, b) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{\frac{1}{b^2} - \frac{1}{a^2}}{b - a} = \dfrac{a^2 - b^2}{a^2 b^2} \cdot \dfrac{1}{b - a} = -\left(\dfrac{a + b}{a^2 b^2}\right)$

$$f(a,b,c) = \frac{f(c,b)-f(b,a)}{c-a} = \frac{-\frac{(b+c)}{b^2c^2}+\frac{(a+b)}{a^2b^2}}{c-a}$$

$$= \frac{1}{b^2}\cdot\frac{1}{c-a}\left[\frac{a+b}{a^2}-\frac{b+c}{c^2}\right] = \frac{1}{b^2}\frac{(ac^2+bc^2-a^2b-a^2c)}{a^2c^2(c-a)}$$

$$= \frac{ac(c-a)+b(c-a)(c+a)}{b^2a^2c^2(c-a)} = \frac{1}{a^2b^2c^2}(ab+bc+ca)$$

Similarly $f(a,b,c,d) = -\frac{(abc+bcd+acd+abd)}{a^2b^2c^2d^2}$

Example 5 : Find $\log_{10} 323.5$ given

x	321.0	322.8	324.2	325.0
$\log_{10} x$	2.50651	2.50893	2.51081	2.51188

Solution :

Divided Difference Table

x	$\log_{10} x$	I order divided difference	II order divided difference	III order divided difference
321	2.50651			
		$\frac{2.50893-2.50651}{322.8-321}=0.00134$		
322.8	2.50893		0	
		$\frac{2.51081-2.50893}{324.2-322.8}=0.00134$		0
324.2	2.51081		0	
		$\frac{2.51188-2.51081}{325.0-324.2}=0.00134$		
325.0	251188			

From the table, $x_0 = 321$; $x_1 = 322.8$; $x_2 = 324.2$; $x_3 = 325.0$,

$f(x_0) = 2.50651$; $f(x_0, x_1) = 0.00134$, $f(x_0, x_1, x_2) = 0$ and $\boxed{x = 323.5}$

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2)$$

$$= 2.50651 + (323.5 - 321)(0.00134) + 0$$

$$= 2.50651 + 0.00334 = 2.50985.$$

Example 6 : The values of y and x are given as below.

x	5	6	9	11
y	12	13	14	16

Find the value of y when $x = 10$.

Solution : From the table :

$x_0 = 5; x_1 = 6; x_2 = 9; x_3 = 11;\ f(x_0) = 12; f(x_0, x_1) = 1;\ f(x_0, x_1, x_2) = -0.167$

$f(x_0, x_1, x_2, x_3) = 0.05$ and $\boxed{x = 10}$

The difference table is

x	y	I order divided difference	II order divided difference	III order divided difference
5	12			
		1		
6	13		−0.167	
		0.333		0.050
9	14		0.133	
		1		
11	16			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$= 12 + (10 - 5)(1) + (10 - 5)(10 - 6)(-0.167) + (10 - 5)(10 - 6)(10 - 9)(0.05)$$

$$= 12 + 5 - 3.340 + 1$$

$$= 14.66$$

$\therefore$ The value of y when $x = 10$ is 14.66

Solution :

Divided Difference Table

x	$f(x)$	I order divided difference	II order divided difference	III order divided difference	IV order divided difference
-4	1245				
		$\frac{33-1245}{-1+4} = -404$			
-1	33		$\frac{-28+404}{0+4} = 94$		
		$\frac{5-33}{0+1} = -28$		$\frac{10-94}{2+4} = -14$	
0	5		$\frac{2+28}{2+1} = 10$		$\frac{13+14}{5+4} = 3$
		$\frac{9-5}{2} = 2$		$\frac{88-10}{5+1} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

from table we observe that $f(x_0) = 1245;\ f(x_0, x_1) = -404$

$f(x_0, x_1, x_2) = 94;\ f(x_0, x_1, x_2, x_3) = -14;\ f(x_0, x_1, x_2, x_3, x_4) = 3$

By Newton's divided difference formula

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &\quad + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \\ &\quad + (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) \\ &= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) \\ &\quad + (x + 4)(x + 1)(x)(-14) + (x + 4)(x + 1)(x)(x - 2)(3) \\ &= 3x^4 - 5x^3 + 6x^2 - 14x - 5 \end{aligned}$$